$$\varepsilon_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \varepsilon_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \alpha = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \delta = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

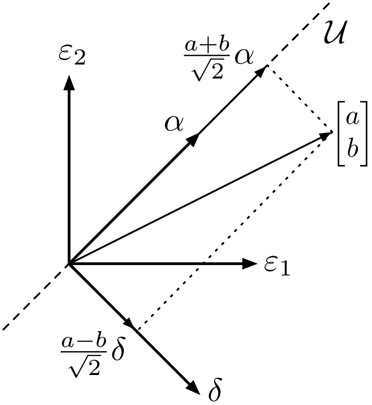

Figure: Optimal approximation in the plane

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\,\varepsilon_1 + b\,\varepsilon_2 = \frac{a+b}{\sqrt{2}}\,\alpha + \frac{a-b}{\sqrt{2}}\,\delta$$

Another basis for $\mathcal{V} = \mathcal{L}^2([0,1))$

- Use step functions for approximation!
- This allows for
  - capturing local properties of functions (*localization*)
  - refinement by adjusting the step width (*resolution*)

  The relevant operations are known as *translation* and *dilation*
- Two basic scaling operations for functions $f : \mathbb{R} \to \mathbb{C}$, in particular for $f \in \mathcal{L}^2(\mathbb{R})$
  - dilation: for $a > 0$

$$(D_a f)(t) = \sqrt{a}\, f(a\, t)$$

  - translation: for $b \in \mathbb{R}$

$$(T_b f)(t) = f(t - b)$$

Illustration of Dilation and Translation (1)

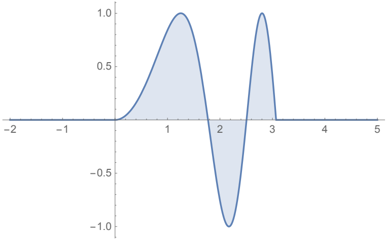

Figure: The function $f(t) = \sin(t^2) \cdot \mathbf{1}_{[0,3\pi)}(t)$

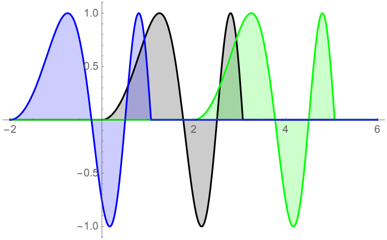

Figure: The functions $f(t)$ (black), $T_2 f(t)$ (green), $T_{-2} f(t)$ (blue)

Illustration of Dilation and Translation (2)

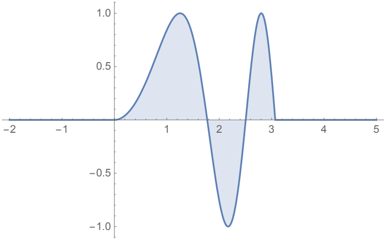

Figure: The function $f(t) = \sin(t^2) \cdot \mathbf{1}_{[0,3\pi)}(t)$

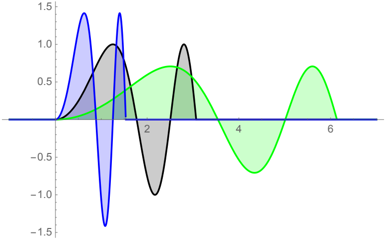

Figure: The functions $f(t)$ (black), $D_{1/2}f(t)$ (green), $D_2f(t)$ (blue)

Illustration of Dilation and Translation (2)

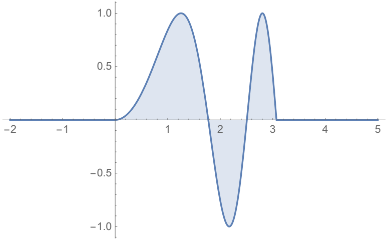

Figure: The function $f(t) = \sin(t^2) \cdot \mathbf{1}_{[0,3\pi)}(t)$

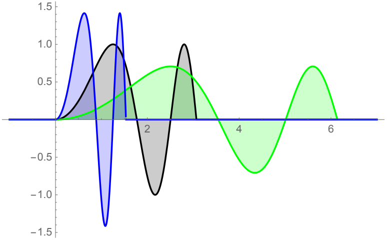

Figure: The functions $f(t)$ (black), $D_{1/2}f(t)$ (green), $D_2 f(t)$ (blue)

Illustration of Dilation and Translation (3)

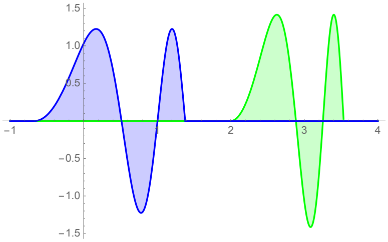

Figure: The functions $T_2 D_2 f(t)$ (green) and $D_{3/2} T_{-1}(t)$ (blue)

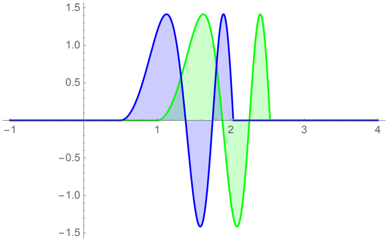

Figure: The functions $T_1 D_{1/2} f(t)$ (green), $D_{1/2} T_1 f(t)$ (blue)

Properties of Dilation and Translation

- Check!

1. $D_a(D_b f) = D_{a\cdot b} f$
2. $T_a(T_b f) = T_{a+b} f$
3. $D_a(T_b f) = T_{b/a}(D_a f)$
4. $\langle f \mid D_a g \rangle = \langle D_{1/a} f \mid g \rangle$
5. $\langle f \mid T_b g \rangle = \langle T_{-b} f \mid g \rangle$
6. $\langle D_a f \mid D_a g \rangle = \langle f \mid g \rangle$, in particular $\|D_a f\| = \|f\|$
7. $\langle T_b f \mid T_b g \rangle = \langle f \mid g \rangle$, in particular $\|T_b f\| = \|f\|$

The Haar scaling function

- For an interval $I = [a, b) \subset \mathbb{R}$ its indicator function is

$$\mathbf{1}_I(t) = \mathbf{1}_{[a,b)}(t) = \begin{cases} 1 & \text{if } \ a \le t < b \\ 0 & \text{otherwise} \end{cases}$$

  Similarly for intervals $[a, b]$ or $(a, b]$ or (a, b)
- The *dyadic itervals* $I_{j,k}$ (for $j, k \in \mathbb{Z}$) are defined as

$$I_{j,k} = [\, k \cdot 2^{-j}, (k+1) \cdot 2^{-j} \,)$$

- The *Haar* scaling function is defined as

$$\phi(t) = \mathbf{1}_{I_{0,0}}(t) = \mathbf{1}_{[0,1)}(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

- For $j, k \in \mathbb{Z}$ put

$$\phi_{j,k}(t) = (D_{2^j} T_k \phi)(t) = 2^{j/2} \cdot \phi(2^j t - k) = 2^{j/2} \mathbf{1}_{I_{j,k}}(t)$$

- j : dilation parameter (*resolution*),
- k : translation parameter (*localization*)

Properties of the $\phi_{j,k}$

- Orthogonality

$$\langle \phi_{j,k} \,|\, \phi_{j,\ell} \rangle = \int_{\mathbb{R}} \phi_{j,k}(t)\, \phi_{j,\ell}(t)\, dt = \delta_{k,\ell}$$

- That is: for any <u>fixed</u> $j \geq 0$ the family

$$\Phi_j = \{\, \phi_{j,k}(t)\,;\, 0 \leq k < 2^j \,\}$$

  is an orthonormal system in $\mathcal{L}^2([0,1))$
- The subspace $\mathcal{V}_j$ of $\mathcal{V} = \mathcal{L}^2([0,1))$ generated by taking Φ_j as its basis is the space of dyadic step functions with step width 2^{-j}

  The space $\mathcal{V}_j$ has dimension 2^j

  This space is known as *approximation subspace* on level j
- The *scaling equation* relates $\mathcal{V}_j$ and $\mathcal{V}_{j+1}$

$$\phi_{j,k}(t) = \frac{1}{\sqrt{2}} \left(\phi_{j+1,2k}(t) + \phi_{j+1,2k+1}(t) \right)$$

Illustrations of the Haar scaling function

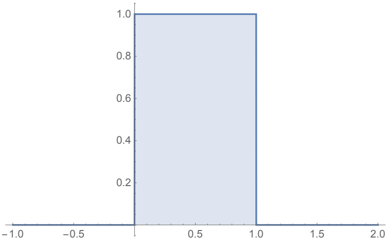

Figure: The Haar scaling function $\phi(t)$

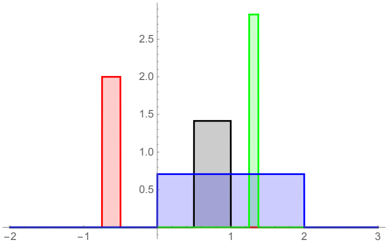

Figure: $\phi_{1,1}(t)$ (black), $\phi_{2,-3}(t)$ (red), $\phi_{3,10}(t)$ (green), $\phi_{-1,0}(t)$ (blue)

Optimal approximation with step functions

- Optimal approximation in $\mathcal{V}_j$ for $f \in \mathcal{L}^2([0,1))$

$$\alpha_j(f;t) = \sum_{0 \le k < 2^j} a_{j,k}\,\phi_{j,k}(t)$$

has *approximation coefficients*

$$a_{j,k} = \langle\, f \mid \phi_{j,k} \,\rangle = 2^{j/2} \int_{I_{j,k}} f(t)\,dt$$

- Important: unlike the Fourier coefficients, the *approximation coefficients* $a_{j,k}$ only depend <u>locally</u> on $f(t)$, precisely:

$$a_{j,k} \cdot \phi_{j,k}(t) = \mu_{j,k}(f) \cdot \mathbf{1}_{I_{j,k}}(t),$$

$$\text{where} \qquad \mu_{j,k}(f) = \frac{1}{|I_{j,k}|} \int_{I_{j,k}} f(t)\,dt$$

is the average of $f(t)$ over $I_{j,k}$

Changing the resolution

- Important question: how do the approximation coefficients $a_{j,k}$ change when changing the resolution parameter j ?
- Partial answer: from $I_{j,k} = I_{j+1,2k} \uplus I_{j+1,2k+1}$ it follows that

$$\begin{aligned}
a_{j,k} &= 2^{j/2} \int_{I_{j,k}} f(t)\,dt \\
&= 2^{j/2} \Big(\int_{I_{j+1,2k}} f(t)\,dt + \int_{I_{j+1,2k+1}} f(t)\,dt \Big) \\
&= \frac{2^{(j+1)/2}}{\sqrt{2}} \Big(\int_{I_{j+1,2k}} f(t)\,dt + \int_{I_{j+1,2k+1}} f(t)\,dt \Big) \\
&= \frac{1}{\sqrt{2}} (a_{j+1,2k} + a_{j+1,2k+1})
\end{aligned}$$

Changing the resolution

- The recurrence equation for the Haar approximation coefficients

$$a_{j,k} = \frac{1}{\sqrt{2}}(a_{j+1,2k} + a_{j+1,2k+1})$$

is really a consequence of the *scaling equation*

$$\phi_{j,k}(t) = \frac{1}{\sqrt{2}}\left(\phi_{j+1,2k}(t) + \phi_{j+1,2k+1}(t)\right),$$

because by linearity of the inner product

$$\langle\, f \mid \phi_{j,k} \,\rangle = \frac{1}{\sqrt{2}}\left(\langle\, f \mid \phi_{j+1,2k} \,\rangle + \langle\, f \mid \phi_{j+1,2k+1} \,\rangle\right)$$

Changing the resolution

- The complete answer:
  - Define *detail coefficients* for $0 \leq k < 2^j$

    $$d_{j,k} = \frac{1}{\sqrt{2}}(a_{j+1,2k} - a_{j+1,2k+1})$$

    then

    $$\begin{bmatrix} a_{j,k} \\ d_{j,k} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_{j+1,2k} \\ a_{j+1,2k+1} \end{bmatrix}$$

    and consequently

    $$\begin{bmatrix} a_{j+1,2k} \\ a_{j+1,2k+1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_{j,k} \\ d_{j,k} \end{bmatrix}$$

- This defines the HAAR transformation at level $j+1$!

$$(a_{j+1,0},\, a_{j+1,1}, \ldots,\, a_{j+1,2^{j+1}-1})$$
$$\updownarrow$$
$$(a_{j,0},\, a_{j,1}, \ldots,\, a_{j,2^j-1}, d_{j,0},\, d_{j,1}, \ldots,\, d_{j,2^j-1})$$

What the $d_{j,k}$ really are

- From the definition:

$$
\begin{aligned}
d_{j,k} &= \frac{1}{\sqrt{2}}(a_{j+1,2k} - a_{j+1,2k+1}) \\
&= \frac{2^{(j+1)/2}}{\sqrt{2}}\left(\int_{I_{j+1,2k}} f(t)\,dt - \int_{I_{j+1,2k+1}} f(t)\,dt\right) \\
&= \langle f \mid \psi_{j,k} \rangle
\end{aligned}
$$

  where $\qquad \psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) \qquad$ and where

$$
\psi(t) = \mathbf{1}_{[0,1/2)}(t) - \mathbf{1}_{[1/2,1)}(t) = \begin{cases} 1 & \text{für } 0 \leq t < 1/2 \\ -1 & \text{für } 1/2 \leq t < 1 \\ 0 & \text{sonst} \end{cases}
$$

  is known as the *Haar wavelet function*

- Note that

$$
\psi_{j,k}(t) = (D_{2^j} T_k \psi)(t)
$$

Illustration of the Haar wavelet function

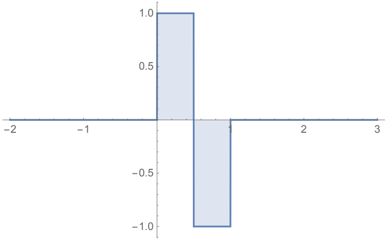

Figure: The Haar wavelet function $\psi(t)$

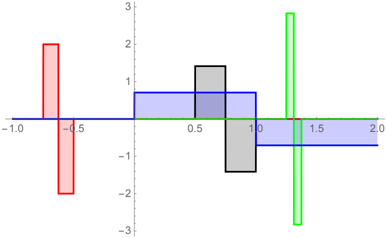

Figure: $\psi_{1,1}(t)$ (black), $\psi_{2,-3}(t)$ (red), $\psi_{3,10}(t)$ (green) $\psi_{-1,0}(t)$ (blue)

The wavelet equation appears

- The definition of the $d_{j,k}$ is equivalent to the *wavelet equation*

$$\psi_{j,k}(t) = \frac{1}{\sqrt{2}} \left(\phi_{j+1,2k}(t) - \phi_{j+1,2k+1}(t)\right)$$

- The family

$$\Psi_j = \{\, \psi_{j,k}(t) \,\}_{0 \le k < 2^j}$$

  is an ONS in $\mathcal{V} = \mathcal{L}^2([0,1))$
- The subspace $\mathcal{W}_j$ of $\mathcal{V} = \mathcal{L}^2([0,1))$ generated by Ψ_j is called *wavelet* or *detail subspace* at level j
- The space $\mathcal{W}_j$ has dimension 2^j
- Check: All $\phi_{j,k}$ are orthogonal to all $\psi_{j',\ell}$ for $j \le j'$ and $(0 \le k < 2^j, 0 \le \ell < 2^{j'})$
- Check: All $\psi_{j,k}$ are orthogonal to all $\psi_{j',\ell}$ for $j' \ne j$

Putting ϕ and ψ together

- The functions

  $$\Phi_{j+1} = \{\, \phi_{j+1,k}(t) \,\}_{0 \le k < 2^{j+1}}$$

  generate (as an ONS) the subspace $\mathcal{V}_{j+1}$ of $\mathcal{V} = \mathcal{L}^2([0,1))$ of step functions of step width 2^{-j-1}

  This space has dimension 2^{j+1}
- By definition

  $$\mathcal{V}_j \subset \mathcal{V}_{j+1} \quad \text{and} \quad \mathcal{W}_j \subset \mathcal{V}_{j+1}$$

- But the space $\mathcal{V}_{j+1}$ also has

  $$\Phi_j \cup \Psi_j = \{\, \phi_{j,k}(t) \,\}_{0 \le k < 2^j} \cup \{\, \psi_{j,k}(t) \,\}_{0 \le k < 2^j}$$

  as an ONS! Hence

  $$\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$$

Two bases in one space

- The 1-level Haar transformation (at level $j+1$) is an orthogonal basis transformation in the space $\mathcal{V}_{j+1}$ between bases
$$\Phi_{j+1} \text{ and } \Phi_j \oplus \Psi_j$$
- which explicitly reads
$$\begin{bmatrix} \phi_{j,k}(t) \\ \psi_{j,k}(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \phi_{j+1,2k}(t) \\ \phi_{j+1,2k+1}(t) \end{bmatrix}$$
and equivalently
$$\begin{bmatrix} \phi_{j+1,2k}(t) \\ \phi_{j+1,2k+1}(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \phi_{j,k}(t) \\ \psi_{j,k}(t) \end{bmatrix}$$

Basic identities

- Haar scaling identity (Analysis)

$$\phi_{j,k}(t) = \frac{1}{\sqrt{2}}(\phi_{j+1,2k}(t) + \phi_{j+1,2k+1}(t))$$

- Haar wavelet identity (Analysis)

$$\psi_{j,k}(t) = \frac{1}{\sqrt{2}}(\phi_{j+1,2k}(t) - \phi_{j+1,2k+1}(t))$$

- Both identities together (Analysis)

$$\begin{bmatrix} \phi_{j,k}(t) \\ \psi_{j,k}(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \phi_{j+1,2k}(t) \\ \phi_{j+1,2k+1}(t) \end{bmatrix}$$

- Reconstruction (Synthesis)

$$\begin{bmatrix} \phi_{j+1,2k}(t) \\ \phi_{j+1,2k+1}(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \phi_{j,k}(t) \\ \psi_{j,k}(t) \end{bmatrix}$$

Transforming the coefficients

- Analysis

$$\begin{bmatrix} a_{j,k} \\ d_{j,k} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_{j+1,2k} \\ a_{j+1,2k+1} \end{bmatrix}$$

- Synthesis

$$\begin{bmatrix} a_{j+1,2k} \\ a_{j+1,2k+1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_{j,k} \\ d_{j,k} \end{bmatrix}$$

- This defines the HAAR transformation at level $j+1$!

$$(a_{j+1,0}, a_{j+1,1}, \ldots, a_{j+1,2^{j+1}-1})$$
$$\updownarrow$$
$$(a_{j,0}, a_{j,1}, \ldots, a_{j,2^j-1}, d_{j,0}, d_{j,1}, \ldots, d_{j,2^j-1})$$

Outlook (for $\mathcal{L}([0,1))$)

- The set of functions

$$\{\,\phi(t)\,\} \cup \bigcup_{j\geq 0} \Psi_j = \{\,\phi(t)\,\} \cup \{\,\psi_{j,\ell}(t)\,;\, j\geq 0, 0\leq \ell < 2^j\,\}$$

  is a Hilbert basis in the space $\mathcal{L}([0,1))$
- This is the HAAR wavelet basis.
- This means that functions $f \in \mathcal{L}^2([0,1))$ can be written as

$$\begin{aligned} f(t) &= \langle\, f(t) \,|\, \phi(t)\,\rangle\,\phi(t) + \sum_{\substack{j\geq 0\\ 0\leq \ell<2^j}} \langle\, f \,|\, \psi_{j,\ell}\,\rangle\,\psi_{j,\ell}(t) \\ &= \int_0^1 f(t)\,dt + \sum_{\substack{j\geq 0\\ 0\leq \ell<2^j}} d_{j,\ell}\,\psi_{j,\ell}(t) \end{aligned}$$

Outlook (for $\mathcal{L}([0,1))$

- For each fixed $J \geq 0$, the set of functions

$$\begin{aligned}\mathcal{H}_J &= \Phi_J \cup \bigcup_{j \geq J} \Psi_j \\ &= \left\{ \phi_{J,k} \,;\, 0 \leq k < 2^J \right\} \cup \left\{ \psi_{j,\ell}(t) \,;\, j \geq J, 0 \leq \ell < 2^j \right\}\end{aligned}$$

  is a Hilbert basis in the space $\mathcal{L}([0,1))$
- This means that functions $f \in \mathcal{L}^2([0,1))$ can be written as

$$\begin{aligned}f(t) &= \sum_{0 \leq k < 2^J} \langle\, f(t) \,|\, \phi_{J,k}(t) \,\rangle\, \phi_{J,k}(t) + \sum_{\substack{j \geq J \\ 0 \leq \ell < 2^j}} \langle\, f \,|\, \psi_{j,\ell} \,\rangle\, \psi_{j,\ell}(t) \\ &= \sum_{0 \leq k < 2^J} a_{J,k}\, \phi_{J,k}(t) + \sum_{\substack{j \geq J \\ 0 \leq \ell < 2^j}} d_{j,\ell}\, \psi_{j,\ell}(t)\end{aligned}$$

Outlook (for $\mathcal{L}(\mathbb{R})$)

- Take intervals $I_{j,k}$ for $j, k \in \mathbb{Z}$
- Take functions $\phi_{j,k}$ and $\psi_{j,k}$ for $j, k \in \mathbb{Z}$
- Define

$$\Phi_j = \{\phi_{j,k}\,;\, k \in \mathbb{Z}\} \qquad \Psi_j = \{\psi_{j,k}\,;\, k \in \mathbb{Z}\}$$

$$\mathcal{H}_J = \Phi_J \cup \bigcup_{j \geq J} \Psi_j \qquad \mathcal{H} = \Phi = \bigcup_{j \geq \mathbb{Z}} \Psi_j$$

$$\mathcal{V}_J = \overline{span}(\Phi_j) \qquad \mathcal{W}_J = \overline{span}(\Psi_j)$$

- Φ_j, Ψ_j, $\mathcal{H}_J$ and $\mathcal{H}$ are orthogonal families
- $\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$ is an orthogonal decomposition
- Scaling and wavelet identities are precisely the same as before
- Coefficient transformations are the same as before
- Haar transformation is the same as before

Outlook (for $\mathcal{L}(\mathbb{R})$)

- For each fixed $J \geq 0$, the set of functions

$$\begin{aligned}\mathcal{H}_J &= \Phi_J \cup \bigcup_{j \geq J} \Psi_j \\ &= \{\, \phi_{J,k} \,;\, k \in \mathbb{Z} \,\} \cup \{\, \psi_{j,\ell}(t) \,;\, j, \ell \in \mathbb{Z} \,\}\end{aligned}$$

  is a Hilbert basis in the space $\mathcal{L}(\mathbb{R})$
- This means that functions $f \in \mathcal{L}^2(\mathbb{R})$ can be written as

$$\begin{aligned}f(t) &= \sum_{k \in \mathbb{Z}} \langle\, f(t) \,|\, \phi_{J,k}(t) \,\rangle\, \phi_{J,k}(t) + \sum_{\substack{j \geq J \\ \ell \in \mathbb{Z}}} \langle\, f \,|\, \psi_{j,\ell} \,\rangle\, \psi_{j,\ell}(t) \\ &= \sum_{k \in \mathbb{Z}} a_{J,k}\, \phi_{J,k}(t) + \sum_{\substack{j \geq J \\ \ell \in \mathbb{Z}}} d_{j,\ell}\, \psi_{j,\ell}(t)\end{aligned}$$

Outlook (for $\mathcal{L}(\mathbb{R})$)

- The set of functions

$$\mathcal{H} = \Psi = \bigcup_{j \in \mathbb{Z}} \Psi_j = \{\, \psi_{j,k}(t)\,;\, j, k \in\in \mathbb{Z} \,\}$$

  is a Hilbert basis in the space $\mathcal{L}(\mathbb{R})$
- This means that functions $f \in \mathcal{L}^2(\mathbb{R})$ can be written as

$$f(t) = \sum_{j,k \in \mathbb{Z}} \langle\, f \mid \psi_{j,k} \,\rangle\, \psi_{j,k}(t) = \sum_{j,k \in \mathbb{Z}} d_{j,k}\, \psi_{j,k}(t)$$