# General considerations

- *Objects* of study are "signals" (either continuous or discrete)
- Mathematically they are represented as functions such as
  - $f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{R} \to \mathbb{C}$
  - $g : \mathbb{Z} \to \mathbb{R}$ or $g : \mathbb{Z} \to \mathbb{C}$
  - $h : [0, 1) \to \mathbb{R}$ or $h : [0, 1) \to \mathbb{C}$
  - Images with $2^8$ grey values can be viewed as functions $k : \mathbb{Z}_N \times \mathbb{Z}_M \to \mathbb{Z}_{256}$ where $\mathbb{Z}_N = \{0, 1, 2, \ldots, N - 1\}$
  - ...
- Intended *high-level actions* on signals are transformations like
  - filtering
  - compression, approximation
  - denoising
  - feature detection
  - fusion, ...
- *Low-level actions* implementing these are: translation, modulation, scaling, addition, multiplication, convolution, ...

# General considerations (contd.)

- Mathematically:
  - signals are elements of appropriate vector spaces (real or complex)
  - actions are (mostly, but not always) linear transformations acting on vectors
- Signal spaces can be endowed with bases of "simple signals", general signals appear as (discrete or continuous) linear combinations of simple signals, e.g. as in

$$f(t) = \sum_k \alpha[k] e_k(t) \quad \text{or} \quad f(t) = \int \alpha(s) e_s(t) ds$$

  where the $e_k(t)$ resp. $e_s(t)$ are simple signals and the $\alpha[k]$ resp. $\alpha(s)$ are constants (w.r.t. the main variable $t$)
- Actions on signals are often realized as actions on the coefficients

# The ideal mathematical context

- In order to have a satisfactory framework for algorithmics the vector spaces of signals need additional *geometric* structure, i.e. concepts like *length*, *distance*, *angle*, *orthogonality*
- The ideal framework (besides of vector spaces of finite dimension) is that of a (separable) *Hilbert space*
  These are vector spaces $\mathcal{H}$ of countable dimension, endowed with a *norm* function $\| \, . \, \| : \mathcal{H} \to \mathbb{R}_+$
  that arises from an inner (scalar) product

$$\langle \, . \, | \, . \, \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

  with the *distance* defined by

$$d(f, g) = \|f - g\|, \text{ where } \|f\| = \sqrt{\langle f | f \rangle}$$

  Orthogonality is defined by

$$f \perp g \quad \Leftrightarrow \quad \langle f | g \rangle = 0$$

  See the Lecture Notes (script) for details

# Important examples

- Finite-dimensional vector spaces $\mathbb{R}^N$ and $\mathbb{C}^N$ with the usual inner product w.r.t. an ON-basis $\mathcal{E} = \{e^1, e^2, \ldots, e^N\}$

$$\boldsymbol{x} = \sum_{i=1}^{N} x_i \, e^i, \;\; \boldsymbol{y} = \sum_{i=1}^{N} y_i \, e^i \quad \Rightarrow \quad \langle \boldsymbol{x} \,|\, \boldsymbol{y} \rangle = \sum_{i=1}^{N} x_i \cdot \overline{y_i}$$

(Note the complex conjugation!)
In particular, the squared (Euclidean) length

$$\|\boldsymbol{x}\|^2 = \langle \boldsymbol{x} \,|\, \boldsymbol{x} \rangle = \sum_{i=1}^{N} |x_i|^2$$

# Important examples (cont.)

- The space $\ell^2$ of bi-infinite discrete signals of finite energy

$$\ell^2 = \{\boldsymbol{x} = (x[i])_{i\in\mathbb{Z}}\,;\, x[i] \in \mathbb{C}, \textstyle\sum_{i\in\mathbb{Z}} |x[i]|^2 < \infty\}$$

  with the inner product

$$\boldsymbol{x} = (x[i])_{i\in\mathbb{Z}},\ \boldsymbol{y} = (y[i])_{i\in\mathbb{Z}} \quad \Rightarrow \quad \langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = \sum_{i\in\mathbb{Z}} x[i] \cdot \overline{y[i]}$$

  so that in particular

$$\|\boldsymbol{x}\|^2 = \langle \boldsymbol{x} \mid \boldsymbol{x} \rangle = \sum_{i\in\mathbb{Z}} |x[i]|^2$$

# Important examples (contd.)

- $\mathcal{L}^2([a,b))$, the space of *square-integrable* functions, i.e., $f : [a,b) \to \mathbb{C}$ over a finite interval $[a,b) \subset \mathbb{R}$ with

$$\int_a^b |f(t)|^2 \, dt < \infty$$

(To be honest the integral must be take in the sense of LEBESGUE, whence the letter $\mathcal{L}$ is used).
The inner product on the space is given by

$$\langle f \mid g \rangle = \int_a^b f(t) \cdot \overline{g(t)} \, dt$$

so that

$$\|f\|^2 = \langle f \mid f \rangle = \int_a^b |f(t)|^2 \, dt$$

# Important examples (contd.)

- $\mathcal{L}^2(\mathbb{R})$, the space of *square-integrable* functions $f : \mathbb{R} \to \mathbb{C}$.
  The definition is as before with
  *a* replaced by $-\infty$ and *b* replaced by $+\infty$
- Important difference to the case of finite intervals is:

  the complex exponentials and trigonometric functions belong obviously to $\mathcal{L}^2(I)$ in the case of finite intervals $I = [a, b)$, but not for the infinite interval $\mathbb{R}$!

# 1-periodic functions

- These are functions $f : \mathbb{R} \to \mathbb{C}$ for which

$$f(t+1) = f(t) \quad \text{for all} \quad t \in \mathbb{R}$$

- They can be viewed as functions $f : [0,1) \to \mathbb{R}$
(or as functions $f : [a, a+1) \to \mathbb{R}$ for any $a \in \mathbb{R}$)
- The family of complex exponentials

$$\omega_m(t) = e^{2\pi imt} \qquad (m \in \mathbb{Z})$$

is an orthonormal family in $\mathcal{L}^2([0,1))$, i.e.,

$$\langle \omega_k(t) \,|\, \omega_\ell(t) \rangle = \int_0^1 e^{2\pi i(k-\ell)t} = \delta_{k,\ell} \qquad (k, \ell \in \mathbb{Z})$$

# 1-periodic functions (contd.)

- Likewise, the trigonometric functions (*harmonics*)

$$\cos(2\pi kt)\ (k \geq 0, k \in \mathbb{Z}), \quad \sin(2\pi \ell t)\ \ (\ell > 0, \ell \in \mathbb{Z})$$

form an orthogonal family in $\mathcal{L}^2([0,1))$ because of

$$\langle \cos(2\pi kt) \mid \cos(2\pi \ell t) \rangle = \begin{cases} 1 & \text{if } k = \ell = 0 \\ 1/2 & \text{if } k = \ell > 0 \\ 0 & \text{if } k \neq \ell \end{cases}$$

$$\langle \sin(2\pi kt) \mid \sin(2\pi \ell t) \rangle = \begin{cases} 1/2 & \text{if } k = \ell > 0 \\ 0 & \text{if } k \neq \ell \end{cases}$$

$$\langle \cos(2\pi kt) \mid \sin(2\pi \ell t) \rangle = 0 \qquad (k \geq 0, \ell > 0)$$

# Fourier's idea (1807)

- Any 1-periodic function can be represented as a linear superposition of complex exponentials resp. of trigonometric functions:

$$f(t) = \frac{a[0]}{2} + \sum_{k>0} a[k]\cos(2\pi kt) + \sum_{\ell>0} b[\ell]\sin(2\pi \ell t)$$
$$= \sum_{m\in\mathbb{Z}} c[m]\,\omega_m(t) = \sum_{m\in\mathbb{Z}} c[m]\, e^{2\pi imt}$$

# FOURIER's idea (contd.)

- Interpretation:
  the *Fourier coefficients* $a[k], b[\ell], c[m]$ tell the intensity (or amplitude) with which the corresponding trigonometric function or exponential is "contained" in the function $f(t)$
- By orthogonality, one expects that

$$c[m] = \langle f(t) \,|\, \omega_m(t) \rangle = \int_0^1 f(t)\, e^{-2\pi i m t} dt$$

- and the *Fourier series expansion* can be written as

$$f(t) = \sum_{m \in \mathbb{Z}} \langle f(t) \,|\, \omega_m(t) \rangle \cdot \omega_m(t)$$

- This is the "blueprint" for many other representations of similar nature
- Similar formulas hold for the $a[k]$ and $b[\ell]$ – see the Lecture Notes or any other text on the subject

# Time domain and frequency domain

- The use of the variable $t$ in $f(t), e^{2\pi imt}$ etc. suggests that one often (but not always) considers $t$ as a time (or space) variable.
  A function $f(t)$ is considered as an object in the *time domain*.
- Parameters $k, \ell, m$ etc. denote frequency (cycles / time unit). In the *frequency domain* an object like $f(t)$ is given by its frequency coefficients $c[m](m \in \mathbb{Z})$ (or $a[k], b[\ell]$)

$$
\begin{aligned}
f(t) &\leftrightarrow (c[m])_{m\in\mathbb{Z}} \\
f(t) &\leftrightarrow (a[k])_{k\geq 0} \cup (b[\ell])_{\ell>0}
\end{aligned}
$$

- The dual nature of signals living in time domain and frequency domain is the fundamental aspect of Fourier theory

# Analysis and synthesis

- The *analysis* formula

$$c[m] = \langle f(t) \,|\, \omega_m(t) \rangle = \int_0^1 f(t)\, e^{-2\pi i m t} dt$$

  shows how the amplitudes $c[m]$ can be computed by correlating the signal $f(t)$ with the basic signals $\omega_m(t)$

- The *synthesis* formula

$$f(t) = \sum_{m \in \mathbb{Z}} c[m]\, \omega_m(t) = \sum_{m \in \mathbb{Z}} c[m]\, e^{2\pi i m t}$$

  shows how the signal $f(t)$ is obtained via superposition of basic signals with the amplitudes as coefficients

# Warning!

- The synthesis formula should be taken here at an intuitive level, as in reality it involves an infinite sum of functions, hence convergence question will show up
- Even if for a given function (signal) $f(t)$ the Fourier coefficients are well defined, it is not at all clear in which sense the syntesis formula is true – if at all
- Making Fourier's idea (arguably one of the most influential ones in all of mathematics) precise turned out to be a major problem in mathematical analysis which kept some of the the best mathematicians busy! It took well over 150 years until a completely satisfactory solution was established – this is a very deep and broad subject with an immense number of applications!

# Big question

- To make things a bit more precise, consider for a given $f(t)$, for which the Fourier coefficients $c[m]$ are well defined, the partial sums

$$S_N(t) = \sum_{m=-N}^{N} c[m] \, e^{2\pi imt}$$

  are approximations of $f(t)$ for an integer $N > 0$
- Each approximation $S_N(t)$ is a finite linear combination of exponentials, hence infinitely differentiable, i.e., as "nice" as a function could possibly be
- The question is:

What happens to $S_N(t)$ as $N \to \infty$ ?

# Classical results

- Two classical results must be mentioned in this context:
  1. Pointwise convergence (DIRICHLET)
     If $f(t)$ is piecewise differentiable, then

$$S_N(t) \to \begin{cases} f(t) & \text{for all } t\in[0,1) \text{ where } f \text{ is continuous} \\ \dfrac{f(t^+)+f(t^-)}{2} & \text{for all } t\in[0,1) \text{ where } f \text{ has a jump discontinuity} \end{cases}$$

  2. $\mathcal{L}^2$-convergence (convergence "in the quadratic mean")
     If $f(t) \in \mathcal{L}^2([0,1))$, then, as $N \to \infty$,

$$d(f, S_N) = \|f(t) - S_N(t)\| = \sqrt{\int_0^1 |f(t) - S_N(t)|^2 dt} \quad \to \quad 0$$

# Side remark

- What has been said about 1-periodic functions carries over to $a$-periodic functions, i.e., functions $f : \mathbb{R} \to \mathbb{C}$ with

$$f(t + a) = f(t) \quad \text{for all} \quad t \in \mathbb{R}$$

- Alternatively, on may regard these as functions $f : [b, b + a) \to \mathbb{C}$ for some $b \in \mathbb{R}$
- The formulas for the Fourier cofficients are obtained by simpe variable transformation from the 1-periodic case, they can be found in the Lecture Notes
- It is often convenient to take either the interval $[0, a)$ or the interval $[-a/2, a/2)$ as domain of definition

# Important example

- The "box" function $f : [-1, +1) \to \mathbb{R}$, given by

$$f(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ 0 & \text{if } 1/2 < |t| \leq 1 \end{cases}$$

- The computation of the Fourier coefficients is an easy exercise, see the Lecture Notes
- The result is a series representation

$$f(t) \text{ ``}{=}\text{''} \ \frac{1}{2} - \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^k \frac{\cos((2k-1)\pi t)}{2k-1}$$

# Details

If one plots the partial sums

$$S_N(t) = \frac{1}{2} - \frac{2}{\pi} \sum_{k=1}^{N} (-1)^k \frac{\cos((2k-1)\pi t)}{2k-1}$$

in the range $0 \leq t \leq 1$ for increasing values of $N$, the one observes the following (see the Mathematica notebook)

1. for $t \neq 1/2$ the value of $S_N(t)$ seems to eventually converge to $f(t) = 1$ if $0 \leq t < 1/2$, or to $f(t) = 0$ if $1/2 < t \leq 1$
2. for $t = 1/2$ one always has $S_N(t) = 1/2$
3. the function $S_N(t)$ heavily oscillates as $t$ approaches the jump discontinuity at $t = 1/2$ of $f(t)$;
   oscillations increase in frequency, as $N$ grows
4. the position of the maximum deviation ("overshooting") of $S_N(t)$ from $f(t)$ moves towards $t = 1/2$ as $N$ grows, but the amount of overshooting does not decrease!
   It remains at about 0.09, independent of $N$

# The Gibbs' phenomenon

- Observations 1.-3. agree with the pointwise convergence theorem, which is no surprise, as $f(t)$ is piecewise differentiable
- The overshooting should not really come as a surprise, since the convergence $S_N(t) \to_{n\to\infty} f(t)$ cannot be uniform – because the limit function is not continuous
- The non-vanishing overshooting bears the name Gibbs phenomenon (or Gibbs-Wilbraham phenomenon) in honor of its discoverers. It is a fundamental property of Fourier series and similarly of the Fourier transform

# Conclusion

- To put the observation of the previous example on a general level, on can state:
  - In Fourier analysis (in the classical sense) one correlates functions to be investigated with complex exponentials (or trigonometric functions), which are functions that are
    - perfectly localized in the frequency domain
    - not at all localized in the time domain
  - Fourier analysis
    - is good for treating *stationary* features of signals
    - it is not so good for analyzing *transient* features (like discontinuities)

# Outlook

Wavelet analysis is a technique designed to overcome these limitations by

- taking as basis functions instead of the complex exponentials functions that are well localized both w.r.t. time and frequency
- generating these basic functions from two “blueprints”, the *scaling function* and the *wavelet function*, by using the operations of *translation* and *dilation*, which leads to the fundamental concept of *multiresolution*

There are many way for constructing the “blueprints”, none of them (except one) is obvious or simple