Basics of X-ray CT: Noise

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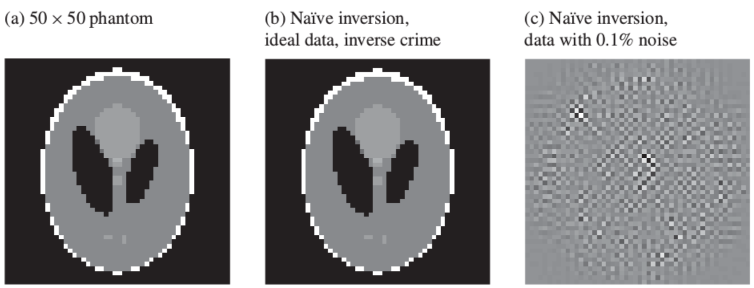
Sarntal, some September 2018



The problem

The Problem: Find f

find me

$$\mathbf{m} = A \, f + e$$

noise

measurement

linear operator

The problem

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find me

$$\mathbf{m} = R\,f + e$$

noise

measurement

linear operator: the Radon transform

The problem

Naive reconstruction:

$$f \quad A^{1}\mathbf{m} \tag{1}$$

The problem

Naive reconstruction: *”inverse crime”*

$$f = A^{-1}\mathbf{m} \tag{1}$$

(a) 50 × 50 phantom

(b) Naïve inversion, ideal data, inverse crime

(c) Naïve inversion, data with 0.1% noise

What do we do?

Impose Occam's razor by imposing a prior distributions.

What do we do?

Impose Occam's razor by imposing a prior distributions.

Aha, right. So what do we actually do?

Overview

Overview I

Tikhonov regularization

- Generalized version
- Computation
- Parameter choice
  - Morozov discrepancy principle
  - L-curve method

Overview II

Total variation regularization

- Comparison to second norm (Tikhonov reg.)
- Computational approaches
  - Quadratic programming
  - Large-scale gradient-based minimization method

Overview III

Curvelet Sparse Regularization

- Curvelet frame
- Computation with ADMM
  - Parameter discussion
- Comparison to TV

Tikhonov regularization

- First choice for linear problems + generalized form accommodates the usage of known properties
- Not edge preserving

$$v = \arg\min_{z \in \mathbb{R}^n} ||Az - m||^2 + a||z||^2 \qquad (2)$$

- Small residual $Av - m$
- v small in L^2 norm (prevents over tting)

Generalized Tikhonov regularization (priori information)

f is close to f :

$$v = \arg\min jj\,Az \quad m jj^2 + a\, jj\, z \quad f \; jj^2 \tag{3}$$

f is known to be smooth:

$$v = \arg\min jj\,Az \quad m jj^2 + a\, jj\, Lz jj^2 \tag{4}$$

or

$$v = \arg\min jj\,Az \quad m jj^2 + a\, jj\, L(z \quad f \;) jj^2 \tag{5}$$

L is a discretized differential operator/matrix

Tikhonov regularization: Computation

Stacked form of the non-generalized equation:

$$\begin{pmatrix} A \\ \sqrt{\alpha}\, I \end{pmatrix} f = \begin{pmatrix} m \\ 0 \end{pmatrix} \qquad (6)$$

written as

$$\tilde{A} f = \tilde{m} \qquad (7)$$

leading to the solution by computing the least-square (no need to compute the SVD):

$$f = \tilde{A} \backslash \tilde{m} \qquad (8)$$

Generalized form:

$$v = (A^T A + \alpha L^T L)^{-1} A^T m \qquad (9)$$

Compute with the conjugate gradient method. (No need to construct the matrices $A; A^T; L; L^T$)

Tikhonov regularization

- Simple implementation
- Problem: How to choose parameter?

Tikhonov regularization: Parameter choice

Morozov discrepancy principle

L-curve method

And other methods e.g. Generalized cross-validation method

Tikhonov regularization: Parameter choice

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Parameter choice: Morozov discrepancy principle

Estimate on error exists => solution with the same level of noise is ok, so choose a, such that

$$||Av - m|| = \text{noise} \tag{10}$$

If

$$||Pm|| <= \text{noise} <= ||m|| \tag{11}$$

then a is unique.

P - orthogonal projection to the subspace Coker(A)

Result

After a simple computation of a longer formula for the deconvolution problem with noise = 11%:

Problem

Morozov discrepancy principle does not apply to the generalized version.

Tikhonov regularization: Parameter choice

Morozov discrepancy principle

L-curve method

And other methods e.g. Generalized cross-validation method

Parameter choice: L-curve method

1. Gather candidates for a
2. Compute the result v for each.
3. Plot $\log(\|Av - m\|); \log(\|Lv\|)$ for results
4. Observe the ”L curve graph”. ”Optimal” solution at the corner.

L-curve method: Example

Total variation regularization

Why? ->

Edge preserving

Total variation regularization

Why? ->

Edge preserving

What? ->

Replace 2-norm by 1-norm in penality term of the generalized Tikhonov regularization

$$\|Az - m\|^2 + \alpha \sum_{j=1}^{n} |(Lz)_j|$$

TV regularization

TV Definition: f is function defined on the interval $[a;b]$. TV is than:

$$TV(f) = \sup \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \quad (12)$$

where the supremum is over all partitions $a = x_0 < x_1 < \dots < x_k = b$ of $[a;b]$

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If differentiable (generalized also to higher dimensions)->

$$TV(f) = \int |\nabla f(x)| dx \tag{13}$$

TV regularization: Edge preserving

Solution is blocker because sharp jumps are not strongly penalized.

$$\|Lf\|_2^2 = 44.44 \qquad \|Lh\|_2^2 = 400$$

$$\|Lf\|_1 = 20 \qquad \|Lh\|_1 = 20$$

TV regularization: Computation

- Medium-scale constrained quadratic programming
- Large-scale gradient-based minimization methods
- And other methods e.g. lagged diffusivity method; Lagrange multiplier methods; frame-based thresholding methods....

TV regularization: Computation with quadratic programming

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$$\text{old:} \quad ||Az - \mu||^2 + \alpha \sum_{j=1}^{n} |(Lz)_j| \quad \text{new:} \quad ||Af||^2 - 2\mu^T Af + \alpha 1^T v_+ + \alpha 1^T v_- \tag{15}$$

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$$\text{old:} \quad \|Az - m\|^2 + \alpha \sum_{j=1}^{n} |(Lz)_j| \quad \text{new:} \quad \|Af\|^2 - 2m^T Af + \alpha 1^T v_+ + \alpha 1^T v_- \tag{15}$$

because $\|Af\|^2 = f^T A^T Af$

$$H := \begin{bmatrix} 2A^T A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad h = \begin{bmatrix} -2A^T m \\ \alpha 1 \\ \alpha 1 \end{bmatrix} \quad y := \begin{bmatrix} f \\ v_+ \\ v_- \end{bmatrix} \tag{16}$$

TV regularization: Computation with quadratic programming

1. Convert the problem to the standard form for quadratic programming.

   ...leading to

$$\arg\min_{y} \frac{1}{2} y^T H y + h^T y \tag{17}$$

   with the constrains

$$L \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ \vdots \\ y_{2n} \end{bmatrix} \begin{bmatrix} y_{2n+1} \\ \vdots \\ y_{3n} \end{bmatrix}$$

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   - the converted problem has 3n degrees of freedom, whereas the original has only n.
   - in the two dimensional case there are 5n.

Result

Parameter choice: S-curve method

Priori information: number of nonzero coef cients in the true signal.

Compute for multiple parameters and choose the one with similar number.

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Gradient descent minimization method of Barzilai and Borwein

$$\arg\min_{f} \|Af - m\|_2^2 + \alpha \|Lf\|_1 = \arg\min_{f} \|Af - m\|_2^2 + \alpha \sum_{i=1}^{k} |f_i - f_{i-1}|$$

Gradient descent minimization method of Barzilai and Borwein

$$\arg\min_f \|Af - m\|_2^2 + \alpha |Lf|_1 = \arg\min_f \|Af - m\|_2^2 + \alpha \sum_{i=1}^{k} |f_i - f_{i-1}|$$

Approximate the L^1 norm with the $L^{1+\epsilon}$:

$$|t|_\epsilon = \sqrt{t^2 + \epsilon} \quad \text{or} \quad |t|_\epsilon = \frac{1}{\epsilon}\log(\cosh(\epsilon t)) \qquad (18)$$

Overview

Shortcoming of TV

- Loss of fine structures and contrast
- May lead to staircasing

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- Loss of ne structures and contrast
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...leading to series expansion frameworks for reconstruction with sparsifying and edge-preserving dictionaries like shearlets and curvelets (for the regularization term)

Problem review

$$\arg\min_{f} \|Af - m\|^2 + \alpha G(f)$$

Sparse regularization

$$\arg\min_{f} ||Af \quad m||^2 + a\,G(f) \tag{19}$$

T is a gradient operator (L) -> TV

T - series expansion framework, e.g. with curvelets -> Curvelet Sparse Regularization

Sparse regularization

Sparse - continuous signal can be represented by a nite number of coef cients in a suitable basis

Sparse Regularization - Tf is supposed to contain relatively few nonzero values

Curvelet Sparse Regularization

Basis: curvelets

T := C the curvelet transform

Minimizing algorithm

Problem: L^1-norm is not continuously differentiable => gradient descent does not work. Options:

1. Approximate L^1 with L^{1+e} (method of Barzilai Borwein with TV)
2. Splitting techniques like Alternating Direction Method of Multipliers (ADMM)

Curvelet Sparse Regularization

$T := C$ the curvelet transform

So lets look at L^1 with curvelets for the penality and ADMM for the minimization.

$T := C$ the curvelet transform

CSR: the curvelet frame

Curvelets: family of functions

location

$$\psi_{j,k,l}$$

scale — orientation

CSR: the curvelet frame

Curvelets: family of functions

$$\psi_{j,k,l}$$

(scale: j, location: k, orientation: l)

So now we can expand any function $f \in L^2(\mathbb{R}^2)$:

$$f = \sum_{j,l,k} \langle \psi_{j,k,l}, f \rangle_{L^2} \psi_{j,k,l}$$

Minimizing algorithm: ADMM

Convert $\arg\min_f ||Af - m||^2 + a|Cf|$

to

$$\arg\min_f ||Af - m||^2 + a|c| \quad \text{s.t.} \quad Cf = c$$

Minimizing algorithm: ADMM

Convert $\arg\min_f \|Af - m\|^2 + \alpha |Cf|$

to

$$\arg\min_f \|Af - m\|^2 + \alpha |c| \quad \text{s.t.} \quad Cf = c$$

and after some fancy stuff (keyword: Lagragian) we arrive at ...

1. A linear inverse problem -> solve approximately with a gradiant method
2. Threshholding step
3. A simple dual update

Minimizing algorithm: ADMM

1. A linear inverse problem -> solve approximately with a gradient method
2. Thresholding step:
   S is the proximity operator to the L^1-norm, "soft-thresholding". Here the threshold being $\frac{a}{b}$
3. A simple dual update

1. $(A^T A + bC^T C)(f^{k+1}) = (A^T m + bC^T(c^k + u^k))$
2. $c^{k+1} = S(C(f^{k+1}) + u^k)$ with $S(x) = \begin{cases} x \quad \text{sgn}(x)\frac{a}{b} & |x| \quad \frac{a}{b} \\ 0 & \text{else} \end{cases}$
3. $u^{k+1} = u^k + C(x^{k+1}) \quad z^{k+1}$

Parameter discussion

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a - our regular regularization parameter

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a - our regular regularization parameter

b - coupling parameter

Parameter discussion

We could try some combinations out and choose the best one.

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How do we know which is ”best”?

Parameter discussion

We could try some combinations out and choose the best one.
How do we know which is ”best”?

We employ a metric to judge the quality of reconstruction.

Parameter discussion

Parameter discussion

Computational expense?

Can we take advantage that ADMM is iterative?

CSR vs TV

CSR > TV structured regions; highly directional, high contrast features with smooth contrast variations

CSR < TV homogeneous regions; CSR oscillating artifacts

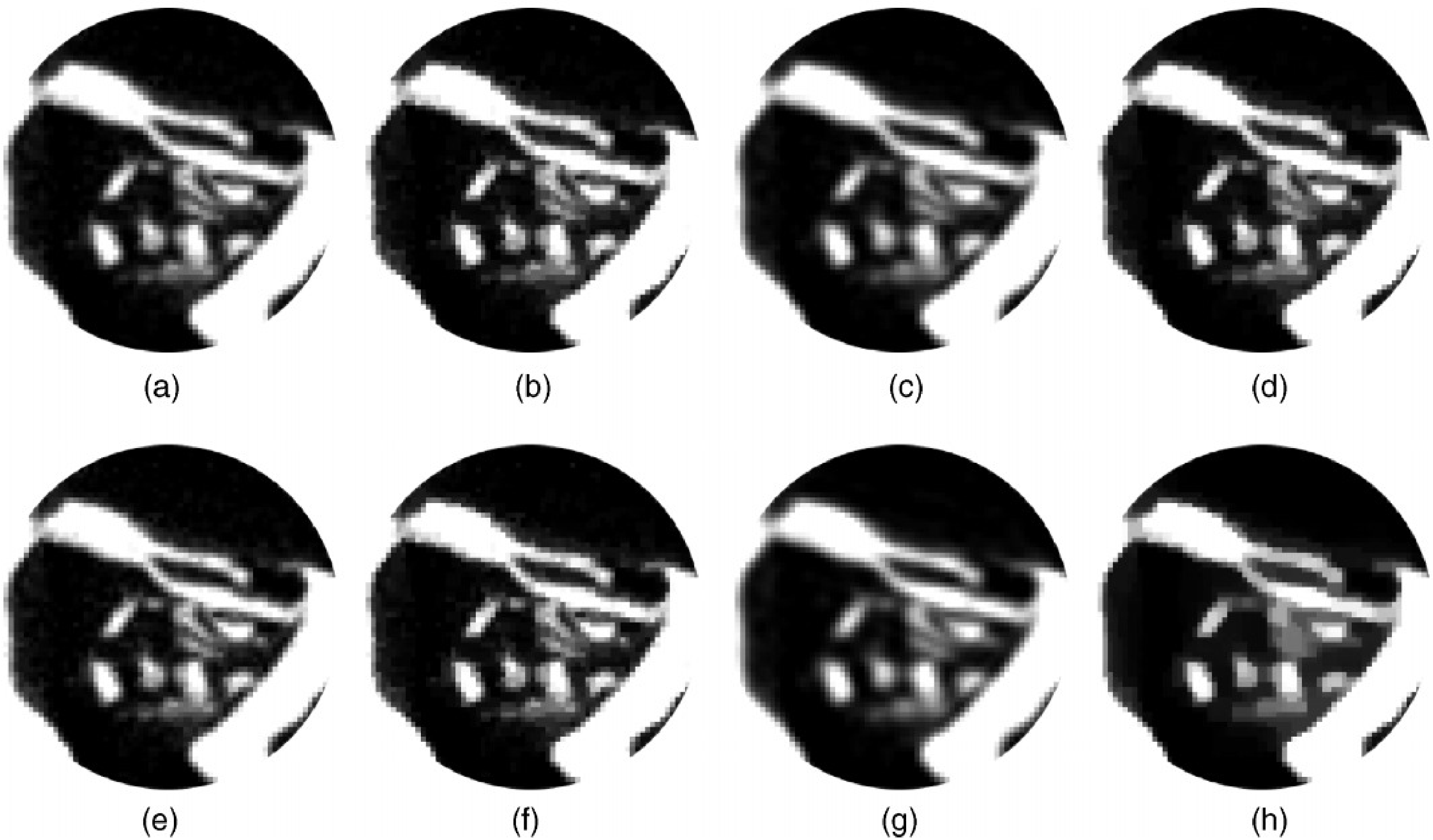
CSR vs TV

CSR > TV structured regions; highly directional, high contrast features with smooth contrast variations

CSR < TV homogeneous regions; CSR oscillating artifacts

Let's see now some real stuff (stuff being a femur mCT)

CSR vs TV



Summary

We must regularize!

We have to make choices about:

- The penality term.
- The parameters' choice.
- The computational approach.

Summary

- Tikhonov reg. favors smooth solutions & uses L^2-norm
- TV & CSR preserve edges & use L^1-norm
  - CSR > TV smooth image with edges along smooth curves
  - CSR < TV homogeneous regions

How do we choose?

We take our knowledge about the expected images and choose the best suitable method for it.

Basics of X-ray CT: Noise

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