Task 2

1-D Transformation using Haar Wavelets

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1 Summary

Functions $f \in L^2[0,1]$ can be represented as

$$f = \sum_{\substack{j \le J \\ 0 \le k < 2^j}} \left\langle f, \psi_{j,k} \right\rangle \psi_{j,k} + \sum_{0 \le k < 2^J} \left\langle f, \phi_{j,k} \right\rangle \phi_{j,k} \tag{1}$$

for all $J \geq 0$. Here, $\phi_{j,k}$, $\psi_{j,k}$ denote orthogonal basis functions. The Haar scaling function $\phi_{j,k}$ is

$$\phi_{j,k} = 2^{j/2} \chi_{I_{j,k}} \quad , \tag{2}$$

and the Haar wavelet function $\psi_{j,k}$ is

$$\psi_{j,k} = 2^{j/2} (\chi_{I^l_{j,k}} - \chi_{I^r_{j,k}}) \quad . \tag{3}$$

Here,

$$I_{j,k} = \left[2^{-j}k, 2^{-j}(k+1)\right) \tag{4}$$

denotes dyadic intervals with the properties that

$$I^l_{j,k} \ = \ I_{j+1,2k} \tag{5}$$

and

$$I^r_{j,k} = I_{j+1,2k+1} \quad , \tag{6}$$

and χ_I denotes the box function with value 1 within the interval I, and 0 everywhere else.

2 1-D Discrete Haar Transformation

2.1 Analysis

In the lecture, we discussed the algorithm for computing the one-dimensional discrete Haar transformation. Implement the analysis filterbank in the function `dht`, following these steps:

1. Define the high pass and low pass filters for analysis.
2. The boundary of the signal may turn out to be problematic when applying a filter. Perpetuate the (to-be-transformed) signal periodically, such that the filters can be applied safely.

3. Convolve the signal with the high pass and low pass filters. For this task, use the matlab function `conv2`.

4. Downsample the coefficients by a factor of 2.

5. Plot a box function, and compute the approximation coefficients and detail coefficients.

6. Vary width and position of the box. What do you observe?

2.2 Synthesis

The original signal can be reconstructed from the wavelet coefficients. Implement a synthesis filterbank in the function `idht`, following these steps:

1. Define the high pass and low pass filters for analysis.

2. Upsample the coefficients by a factor of 2.

3. Perpetuate this signal periodically, such that it matches the periodic continuation of the analysis step.

4. Convolve this signal with the high pass and low pass filters, and add the resulting signals.

5. Plot the reconstructed signal and compare it to the original signal.

3 1-D Haar Multi-Resolution Analysis

The approximation coefficients of the signal can further be decomposed. Iterating this procedure splits the signal into multiple frequency bands. This process is called multi-resolution analysis. Implement the iterated filterbanks for analysis and synthesis:

1. For the wavelet analysis, compute n iterations of `dht`. Plot the detail coefficients and approximation coefficients of every iteration.

2. Add the resulting detail coefficients from each step to the vector of wavelet coefficients.

3. After the last iteration step, add the remaining approximation coefficients to the vector of wavelet coefficients.

4. Insert a visualization of the current set of wavelet coefficients to each iteration. In order to reuse the same Figure, use the matlab command `waitforbuttonpress`. This command shall interrupt the wavelet analysis until a key is pressed, to allow user inspection of the wavelet coefficients.

5. Compute the reconstruction of the original signal from n iteration. Your method should make use of the function `idht`.

6. Examine the decomposition of the box function.

7. Examine the decomposition of the function

$$f(t) = \cos(k \cdot 2\pi \cdot t) \tag{7}$$

on the interval $0 \leq t < 1$ for different values of k.

8. Compute the energy

$$E_i = \sum_t |d_i[t]|^2 \tag{8}$$

of the detail coefficients $d_i[t]$ for each decomposition level $i \leq n$ n, and plot them. What do you observe for different frequencies of the cosine oscillation? What do you observe for linear combinations of different cosine oscillations?

9. Add a threshold s in order to set all detail coefficients to 0 whose absolute value is lower than s.

10. Compute the function

$$g(t) = \cos(8\pi \cdot t) + 0.2\cos(100\pi \cdot t). \tag{9}$$

What do you observe for increasing threshold levels s?

4 Follow-up on Sawtooth and Triangle

If $f : \mathbb{R} \to \mathbb{R}$ is a real a-periodic function and

$$S^f(t) = \sum_{n\in\mathbb{Z}} c_f[n]\, e^{2\pi int/a}$$

its Fourier series, written in the basis of complex exponentials, consider the following operation on its Fourier coefficients:

$\mathcal{H}$: replace $c_f[n]$ by $c_{\mathcal{H}f} = i \cdot \mathrm{sgn}(n) \cdot c_f[n]$ for all $n \in \mathbb{Z}$.

Then one can talk about a hypothetical new function $\mathcal{H}f$ which has as its Fourier series

$$S^{\mathcal{H}f}(t) = \sum_{n\in\mathbb{Z}} c_{\mathcal{H}f}[n]\, e^{2\pi int/a}$$

together with the corresponding partial sums (for $N \in \mathbb{N}$)

$$S_N^{\mathcal{H}f}(t) = \sum_{n=-N}^{N} c_{\mathcal{H}f}[n]\, e^{2\pi int/a}.$$

1. Why do the $S_n^{\mathcal{H}f}(t)$ also have real values (so that the limit $\mathcal{H}f(t)$ will also be an a-periodic real function) ?

2. What is the effect of passing from $f(t)$ to $\mathcal{H}f(t)$ when writing the Fourier series in the trigonometric basis ?

3. Plot the approximations $S_N^{\mathcal{H}f}(t)$ for the functions *triangle*(t) and *sawtooth*(t) of Problem 3 for increasing values of N. What do you observe ? What happens at the jump discontinuity $t = \pi$ for *sawtooth*(t) ? Can you find out the reason ?