Some essential linear algebra

- $\mathcal{V}$: a complex (or real) vector space
  $\langle\, \boldsymbol{u} \,|\, \boldsymbol{v} \,\rangle$ an inner (scalar) product on $\mathcal{V}$
  $\|\, \boldsymbol{u} \,\|$ the norm (length) given by $\sqrt{\langle\, \boldsymbol{u} \,|\, \boldsymbol{u} \,\rangle}$
  $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$ the distance (metric) induced by the inner product
  $\langle\, \boldsymbol{u} \,|\, \boldsymbol{v} \,\rangle\, \boldsymbol{v}$: projection of $\boldsymbol{u}$ on the line def. by $\boldsymbol{v}$ (if $\|\boldsymbol{v}\| = 1$)
- important properties (inequalities)
  1. CAUCHY-SCHWARZ $|\langle\, \boldsymbol{u} \,|\, \boldsymbol{v} \rangle| \leq \|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\|$
  2. MINKOWSKI $\|\, \boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$
  3. Parallelogram $\|\, \boldsymbol{u} + \boldsymbol{v}\|^2 + \|\, \boldsymbol{u} - \boldsymbol{v}\|^2 \leq \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$
- Definition: A family of vectors $\mathcal{E} = \{\boldsymbol{e}^1, \boldsymbol{e}^2, \ldots\}$ is
  - an orthogonal system (OS) in $\mathcal{V}$ if $\langle \boldsymbol{e}^k, \boldsymbol{e}^\ell \rangle = 0$ for $k \neq \ell$
  - an ortho<u>normal</u> system (ONS) in $\mathcal{V}$ if $\langle \boldsymbol{e}^k, \boldsymbol{e}^\ell \rangle = \delta_{k,\ell}$ for $k \neq \ell$

The finite-dimensional case

- For $\mathcal{V}$ finite-dimensional $\mathcal{V}$ with orthonormal basis $\mathcal{E} = \{\boldsymbol{e}^1, \boldsymbol{e}^2, \ldots, \boldsymbol{e}^n)$, the standard inner product is given by

$$\begin{aligned} \langle\, \boldsymbol{u} \,|\, \boldsymbol{v} \,\rangle &= \langle\, \textstyle\sum_{k=1}^n u_k\, \boldsymbol{e}^k \,|\, \sum_{k=1}^n v_\ell\, \boldsymbol{e}^\ell \,\rangle \\ &= \textstyle\sum_{k=1}^n \sum_{\ell=1}^n u_k v_\ell \,\langle\, \boldsymbol{e}^k \,|\, \boldsymbol{e}^\ell \,\rangle \\ &= \textstyle\sum_{k=1}^n u_k \cdot \overline{v_k} \end{aligned}$$

- In terms of $\mathcal{E}$ one has

base $\mathcal{E}$ expansion	$\boldsymbol{u} = \sum_{k=1}^n \langle\, \boldsymbol{u} \,\vert\, \boldsymbol{e}^k \,\rangle\, \boldsymbol{e}^k$ i.e. $u_k = \langle\, \boldsymbol{u} \,\vert\, \boldsymbol{e}^k \,\rangle$
inner product	$\langle\, \boldsymbol{u} \,\vert\, \boldsymbol{v} \,\rangle = \sum_{k=1}^n \langle\, \boldsymbol{u} \,\vert\, \boldsymbol{e}^k \,\rangle \langle\, \boldsymbol{e}^k \,\vert\, \boldsymbol{v} \,\rangle$
norm (length)	$\Vert\boldsymbol{u}\Vert^2 = \sum_{k=1}^n \vert\langle\, \boldsymbol{u} \,\vert\, \boldsymbol{e}^k \,\rangle\vert^2$

- Geometrically:
  $u_k = \langle\, \boldsymbol{u} \,|\, \boldsymbol{e}^k \,\rangle\, \boldsymbol{e}^k =$ projection of $\boldsymbol{u}$ onto the line defined by $\boldsymbol{e}^k$

Change of basis

- If $\mathcal{F} = \{\boldsymbol{f}^1, \boldsymbol{f}^2, \dots, \boldsymbol{f}^n\}$ is another ONS of $\mathcal{V}$ w.r.t. $\langle\, . \,|\, . \,\rangle$, then

$$\boldsymbol{f}^k = \sum_{1 \le j \le n} \langle\, \boldsymbol{f}^k \,|\, \boldsymbol{e}^j \,\rangle\, \boldsymbol{e}^j \quad \text{and} \quad \boldsymbol{e}^j = \sum_{1 \le k \le n} \langle\, \boldsymbol{e}^j \,|\, \boldsymbol{f}^k \,\rangle\, \boldsymbol{f}^k$$

- Here $U = \Big[\langle\, \boldsymbol{e}^j \,|\, \boldsymbol{f}^k \,\rangle\Big]_{1 \le j,k \le n}$ is a unitary matrix, i.e.,

$$U^{-1} = \Big[\langle\, \boldsymbol{f}^k \,|\, \boldsymbol{e}^j \,\rangle\Big]_{1 \le k,j \le n} = \Big[\overline{\langle\, \boldsymbol{e}^j \,|\, \boldsymbol{f}^k \,\rangle}\Big]_{1 \le k,j \le n} = U^\dagger$$

  $U^\dagger$ is the conjugate-transpose of U (also called *adjoint*)
- Transformation of the coefficients

$$\langle\, \boldsymbol{u} \,|\, \boldsymbol{e}^j \,\rangle = \textstyle\sum_{k=1}^n \langle\, \boldsymbol{u} \,|\, \boldsymbol{f}^k \,\rangle \langle\, \boldsymbol{f}^k \,|\, \boldsymbol{e}^j \,\rangle \qquad (1 \le j \le n)$$
$$\langle\, \boldsymbol{u} \,|\, \boldsymbol{f}^k \,\rangle = \textstyle\sum_{j=1}^n \langle\, \boldsymbol{u} \,|\, \boldsymbol{e}^j \,\rangle \langle\, \boldsymbol{e}^j \,|\, \boldsymbol{f}^k \,\rangle \qquad (1 \le k \le n)$$

Very important example: the Discrete Fourier Transform

- $\mathcal{V} = \mathbb{C}^N$ with its usual inner product
- the standard basis $\mathcal{E}_N$

$$\boldsymbol{e}^j = (0, \ldots, 0, 1, 0, \ldots, 0)^t \quad (0 \leq j < N)$$

- the DFT-basis $\mathcal{F}_N$ with $\omega_N = e^{2\pi i/N}$

$$\begin{aligned}\boldsymbol{f}^j &= \frac{1}{\sqrt{N}} \left(1 \, , \, \omega_N^j \, , \, (\omega_N^j)^2 \, , \, \ldots , (\omega_N^j)^{N-1} \right)^t \quad (0 \leq j < N) \\ &= \frac{1}{\sqrt{N}} \left(\omega_N^{j \cdot 0} \, , \, \omega_N^{j \cdot 1} \, , \, \omega_N^{j \cdot 2} \, , \, \ldots \, , \, \omega_N^{j \cdot (N-1)} \right)^t\end{aligned}$$

- the DFT-matrix U_N and its inverse

$$U_N = \frac{1}{\sqrt{N}} \left[\omega_N^{j \cdot k} \right]_{0 \leq j,k < N} \qquad U_N^{-1} = \frac{1}{\sqrt{N}} \left[\omega_N^{-j \cdot k} \right]_{0 \leq j,k < N}$$

DFT_4 and DFT_6

- DFT_4

$$U_4 = \frac{1}{2}\begin{bmatrix} i^0 & i^0 & i^0 & i^0 \\ i^0 & i^1 & i^2 & i^3 \\ i^0 & i^2 & i^4 & i^6 \\ i^0 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i^0 & i^0 & i^0 & i^0 \\ i^0 & i^1 & i^2 & i^3 \\ i^0 & i^2 & i^0 & i^2 \\ i^0 & i^3 & i^2 & i^1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

- DFT_6

$$U_6 = \frac{1}{\sqrt{6}}\left[\ \omega_6^{k\cdot\ell}\ \right]_{0\le k,\ell<6} = \frac{1}{\sqrt{6}}\begin{bmatrix} \omega_6^0 & \omega_6^0 & \omega_6^0 & \omega_6^0 & \omega_6^0 & \omega_6^0 \\ \omega_6^0 & \omega_6^1 & \omega_6^2 & \omega_6^3 & \omega_6^4 & \omega_6^5 \\ \omega_6^0 & \omega_6^2 & \omega_6^4 & \omega_6^0 & \omega_6^2 & \omega_6^4 \\ \omega_6^0 & \omega_6^3 & \omega_6^0 & \omega_6^3 & \omega_6^0 & \omega_6^3 \\ \omega_6^0 & \omega_6^4 & \omega_6^2 & \omega_6^0 & \omega_6^4 & \omega_6^2 \\ \omega_6^0 & \omega_6^5 & \omega_6^4 & \omega_6^3 & \omega_6^2 & \omega_6^1 \end{bmatrix}$$

ω_6^0	ω_6^1	ω_6^2	ω_6^3	ω_6^4	ω_6^5
1	$\frac{1+i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	-1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{1-i\sqrt{3}}{2}$

DFT_7

- DFT_7

$$\begin{pmatrix} 0.378 & 0.378 & 0.378 & \dots & 0.378 \\ 0.378 & 0.236+0.296i & -0.084+0.368i & \dots & 0.236-0.296i \\ 0.378 & -0.084+0.368i & -0.341-0.164i & \dots & -0.084-0.368i \\ 0.378 & -0.341+0.164i & 0.236-0.296i & \dots & -0.341-0.164i \\ 0.378 & -0.341-0.164i & 0.236+0.296i & \dots & -0.341+0.164i \\ 0.378 & -0.084-0.368i & -0.341+0.164i & \dots & -0.084+0.368i \\ 0.378 & 0.236-0.296i & -0.084-0.368i & \dots & 0.236+0.296i \end{pmatrix}$$

$$\omega_7 = e^{2\pi i/7} = 0.62349\ldots + 0.781831\ldots i$$

$$\frac{1}{\sqrt{7}}\omega_7 = \frac{1}{\sqrt{7}}e^{2\pi i/7} = 0.235657\ldots + 0.295505\ldots i$$

Orthogonal transforms

Other important orthogonal transforms used in image processing:

- DCT : Discrete Cosine Transform
- HWT : Hadamard-Walsh Transform
- KLT : Karhunen-Loève Transform
- DWT : Discrete Wavelet Transform

Optimal approximation: The projection theorem

- Theorem
  $\mathcal{V}$: a vector space with inner product $\langle\,.\,|\,.\,\rangle$ and norm $\|\,.\,\|$
  $\mathcal{U}$: a finite-dimensional subspace of $\mathcal{V}$
  $\{\boldsymbol{e}^1, \boldsymbol{e}^2, \ldots, \boldsymbol{e}^n\}$ an orthonormal basis of U
  Then:
  For each $\boldsymbol{v} \in \mathcal{V}$ there exists a unique element $\boldsymbol{u}_v \in \mathcal{U}$ which minimizes the distance $d(\boldsymbol{v}, \boldsymbol{u}) = \|\boldsymbol{v} - \boldsymbol{u}\|$ $(\boldsymbol{u} \in \mathcal{U})$.
  This element is

$$(*) \quad \boldsymbol{u}_v = \sum_{k=1}^{n} \langle\, \boldsymbol{v} \,|\, \boldsymbol{e}^k \,\rangle\, \boldsymbol{e}^k, \qquad \begin{cases} \text{the } \textit{orthogonal projection} \\ \text{of } \boldsymbol{v} \text{ onto } \mathcal{U} \end{cases}$$

  and the decomposition of $\boldsymbol{v}$ is a unique

$$\boldsymbol{v} = \underbrace{\boldsymbol{v} - \boldsymbol{u}_v}_{\in \mathcal{U}^\perp} + \underbrace{\boldsymbol{u}_v}_{\in \mathcal{U}}$$

Optimal approximation: The projection theorem

- Proof.
  Define $\boldsymbol{u}_v$ as in $(*)$. Then for $1 \leq \ell \leq n$

$$\begin{aligned}\langle\, \boldsymbol{v} - \boldsymbol{u}_v \,|\, \boldsymbol{e}^\ell \,\rangle &= \langle\, \boldsymbol{v} - \textstyle\sum_{k=1}^n \langle\, \boldsymbol{v} \,|\, \boldsymbol{e}^k \,\rangle\, \boldsymbol{e}^k \,|\, \boldsymbol{e}^\ell \rangle \\ &= \langle\, \boldsymbol{v} \,|\, \boldsymbol{e}^\ell \rangle - \textstyle\sum_{k=1}^n \langle\, \boldsymbol{v} \,|\, \boldsymbol{e}^k \,\rangle \langle\, \boldsymbol{e}^k \,|\, \boldsymbol{e}^\ell \,\rangle = 0\end{aligned}$$

  that is: $\boldsymbol{v} - \boldsymbol{u}_v \in \mathcal{U}^\perp$
  If $\boldsymbol{u} \in \mathcal{U}$ is any element, then $\boldsymbol{u} - \boldsymbol{u}_v \in \mathcal{U}$, hence

$$\langle\, \boldsymbol{v} - \boldsymbol{u}_v \,|\, \boldsymbol{u} - \boldsymbol{u}_v \,\rangle = 0$$

  But (Pythagoras!)

$$\|\boldsymbol{v} - \boldsymbol{u}\|^2 = \|\boldsymbol{v} - \boldsymbol{u}_v\|^2 + \|\boldsymbol{u}_v - \boldsymbol{u}\|^2 \geq \|\boldsymbol{v} - \boldsymbol{u}_v\|^2$$

  with equality if and only if $\boldsymbol{u} = \boldsymbol{u}_v$ $\square$

Another important consequence

(same scenario as before)

- BESSEL's inequality

  For $\boldsymbol{v} \in \mathcal{V}$ and any $N \geq 0$ with $v_N = \sum_{k=1}^{N} \langle\, \boldsymbol{v} \mid \boldsymbol{e}^k \,\rangle\, \boldsymbol{e}^k$, then

  $$\|\boldsymbol{v}_N\|^2 = \sum_{k=1}^{N} |\langle\, \boldsymbol{v} \mid \boldsymbol{e}^k \,\rangle|^2 \leq \|\boldsymbol{v}\|^2$$

  because $\boldsymbol{v} - \boldsymbol{v}_N \perp \{\boldsymbol{e}^1, \ldots, \boldsymbol{e}^N\}$

What is a Hilbert space?

- $\mathcal{H}$: vector space with scalar product $\langle\, . \,|\, . \,\rangle$, norm $\|\, . \,\|$
  $\mathcal{E} = \{\boldsymbol{e}^0, \boldsymbol{e}^1, \ldots\} = \{\boldsymbol{e}^n\}_{n\in\mathbb{N}}$ an ONS in $\mathcal{H}$
  $\mathcal{F}$ = subspace of all finite linear combinations of elements of $\mathcal{E}$
- Theorem: The following properties are equivalent
  1. For all $\boldsymbol{u} \in \mathcal{H}$, if $\boldsymbol{u}_N = \sum_{k=0}^{N} \langle\, \boldsymbol{u} \,|\, \boldsymbol{e}^k \,\rangle \boldsymbol{e}^k$, then

     $$\lim_{N\to\infty} \|\boldsymbol{u} - \boldsymbol{u}_N\| = 0$$

     This is written as $\boldsymbol{u} = \sum_{k=0}^{\infty} \langle\, \boldsymbol{u} \,|\, \boldsymbol{e}^k \,\rangle\, \boldsymbol{e}^k$
  2. For all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}$:

     $$\langle\, \boldsymbol{u} \,|\, \boldsymbol{v} \,\rangle = \sum_{k=0}^{\infty} \langle\, \boldsymbol{u} \,|\, \boldsymbol{e}^k \,\rangle \langle\, \boldsymbol{e}^k \,|\, \boldsymbol{v} \,\rangle$$

  3. For all $\boldsymbol{u} \in \mathcal{H}$

     $$\|\boldsymbol{u}\|^2 = \sum_{k=0}^{\infty} |\langle\, \boldsymbol{u} \,|\, \boldsymbol{e}^k \,\rangle|^2$$

What is a Hilbert space?

- Theorem (ctd.)
  4. For all $\boldsymbol{u} \in \mathcal{H}$
  $$\text{if } \langle \boldsymbol{u} \,|\, \boldsymbol{e}^k \rangle = 0 \text{ for all } k \in \mathbb{N}, \text{ then } \boldsymbol{u} = 0$$
  5. $\mathcal{F}$ is *dense* in $\mathcal{H}$, i.e.
  $$\text{for any } \boldsymbol{u} \in \mathcal{H}, \varepsilon > 0 \text{ there is a } \boldsymbol{f} \in \mathcal{F} \text{ such that } \|\boldsymbol{u} - \boldsymbol{f}\| < \varepsilon$$
  6. $\mathcal{F}^{\perp} = \{0\}$

  If these properties hold, $\mathcal{H}$ is called a (separable) *Hilbert space*, and $\mathcal{E}$ is a *Hilbert basis* of $\mathcal{H}$
- Examples are the spaces ℓ^2, $\mathcal{L}^2([0, a))$, $\mathcal{L}^2(\mathbb{R})$ of square-summable sequences and square-integrable functions

The examples

- ℓ^2, the space of square summable sequences, has (among others) the Hilbert basis of "unit vectors"

$$\boldsymbol{\delta}_k = (\delta_{k,j})_{j\in\mathbb{Z}} \quad (k \in \mathbb{Z})$$

- $\mathcal{L}^2([0,a))$, the space of square-integrable functions over a <u>finite</u> interval $[0,a)$ has (among others) the Hilbert basis of complex exponentials

$$\omega_k(t) = \frac{1}{a} e^{2\pi i k t/a} \qquad (k \in \mathbb{Z})$$

  or of the harmonics

$$\frac{1}{a}\cos(2\pi k t/a) \ \ (k \in \mathbb{N}) \ \ \text{and} \ \ \frac{1}{a}\sin(2\pi \ell t/a) \ \ (\ell \in \mathbb{N}_{\geq 0})$$

- A Hilbert basis of the space $\mathcal{L}^2(\mathbb{R})$ of square-integrable functions over $\mathbb{R}$ is not obvious!
  Such bases will appear naturally in Wavelet theory!
- From an algebraic point of view all these spaces are "the same" (i.e., they are *isomorphic*)

Computing in Hilbert bases

- If $\mathcal{E} = \{\boldsymbol{e}^k\}_{k\in\mathbb{N}}$ is a Hilbert basis of $\mathcal{H}$, then for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}$

1. generalized Fourier expansion:

$$\boldsymbol{u} = \sum_{k\in\mathbb{N}} \langle\, \boldsymbol{u} \,|\, \boldsymbol{e}^k \,\rangle\, \boldsymbol{e}^k$$

2. inner product

$$\langle\, \boldsymbol{u} \,|\, \boldsymbol{v} \,\rangle = \sum_{k\in\mathbb{N}} \langle\, \boldsymbol{u} \,|\, \boldsymbol{e}^k \,\rangle \langle\, \boldsymbol{e}^k \,|\, \boldsymbol{v} \,\rangle$$

3. norm (length, energy)

$$\|\boldsymbol{u}\|^2 = \sum_{k\in\mathbb{N}} |\langle\, \boldsymbol{u} \,|\, \boldsymbol{e}^k \,\rangle|^2$$

. . . The best of all possible worlds . . .