SVD: Singular Value Decomposition

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SVD is a matrix factorization technique.
It can be applied on any real or complex $m \times n$ matrix.
It is often used in computing the pseudo-inverse of a matrix and in determining the rank, range and null space of a matrix.

Spectral theorem

Let A be a normal matrix.
If A is a real matrix, then A is normal if $A^T A = AA^T$, where A^T is the transpose of A.
If A is a complex matrix, then A is normal if $A^* A = AA^*$ where A^* is the conjugate transpose of A.
All normal matrices are square.
Theorem: A can be unitarily diagonalized using a basis of eigenvectors.

$$D = P^{-1} A P \tag{1}$$

where P is the matrix with As eigenvectors as its columns, and D is a diagonal matrix of eigenvalues.

SVD: a generalization of the spectral theorem

Let M be an $m \times n$ matrix.
Then M can be factorized as follows:

$$M = USV^T \text{ or } M = USV^* \text{if } M \text{ is complex} \tag{2}$$

where

$-$ U is an $m \times m$ orthogonal matrix (unitary if U is complex),

- S is an $m \times n$ matrix with nonnegative numbers on the diagonal (as defined for a rectangular matrix) and zeros off the diagonal, and
- V^T is another $n \times n$ orthogonal matrix.

This type of factorization is called the *Singular-Value Decomposition (SVD)* of M. This decomposition shows that the function of every matrix can be described as a rotation, followed by a stretch, followed by another rotation.

Note that

- V contains a set of orthonormal "input" or "analysing" basis vector directions for M. The columns of V are the right-singular vectors of M.
- U contains a set of orthonormal "output" basis vector directions for M. The columns of U are the left-singular vectors of M.
- S contains the singular values, which can be thought of as scalar "gain controls" by which each corresponding input is multiplied to give a corresponding output.

Applications

1. Pseudoinverse
Let M^+ be the pseudoinverse of M.
To compute the pseudoinverse:

a. Do SVD(M) to compute U, S, and V.

b. Let S^+ be the pseudo-inverse of S, which is formed by transposing S and replacing every nonzero (i.e. diagonal) entry by its reciprocal.

c. $M^+ = VS^+U$

The pseudoinverse is used in solving linear least squares problems.

2. Rank of M
The rank of M equals the number of non-zero singular values which is the same as the number of non-zero elements in S. The rank of a matrix M is the number of linearly independent rows (or columns, it is equivalent) of M.

3. Null space of M
The null space of M is the set of vectors $\boldsymbol{x}$ for which $M\boldsymbol{x} = 0$. The right singular vectors of M, i.e. the column vectors of V, which correspond to the non-zero singular values of M, span the null space of M.

4. Range of M
The range of M is the set of vectors $\boldsymbol{b}$ for which $M\boldsymbol{x} = \boldsymbol{b}$ has a solution for $\boldsymbol{x}$. The left singular vectors of M, i.e. the column vectors of U, which correspond to the non-zero singular values of M, span the range of M.

SVD and Eigenvalue Decomposition

In the special case that M is a *Hermitian matrix* (i.e. a normal matrix which is equal to its own conjugate transpose, $M = M^*$), then the singular values and singular vectors coincide with the eigenvalues and eigenvectors of M,

$$M = VSV^*, \tag{3}$$

where the columns of V are the eigenvectors of M, and S is a diagonal matrix of eigenvalues.

Linear Least Squares

We have m linear equations with n unknowns, where $m > n$.
In a matrix form:

$$A\boldsymbol{x} = \boldsymbol{b} \tag{4}$$

where A is a known $m \times n$ matrix, $\boldsymbol{x}$ is an n-element vector of unknowns and $\boldsymbol{b}$ is an m-element vector of measurements.
We want to minimize the Euclidean norm squared of the residual $A\boldsymbol{x} - \boldsymbol{b}$,

$$\min \|A\boldsymbol{x} - \boldsymbol{b}\|^2. \tag{5}$$

Then the minimizing vector $\hat{\boldsymbol{x}}$ is:

$$\hat{\boldsymbol{x}} = A^+\boldsymbol{b} \tag{6}$$

where A^+ is the pseudoinverse, $A^+ = (A^TA)^{-1}A^T$.