Polynomial Classifiers

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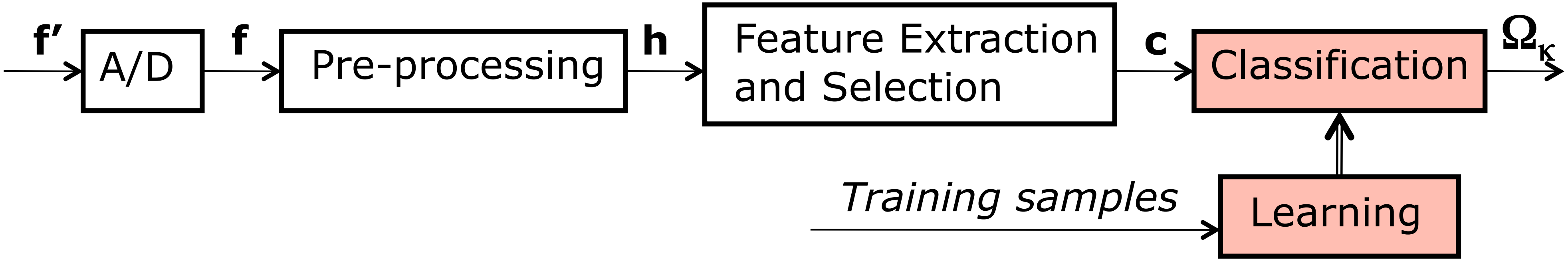
Pattern Recognition Pipeline



- Classification
  - Statistical classifiers
    - Bayesian classifier
    - Gaussian classifier
  - Polynomial classifiers

Key Concepts of Polynomial Classifiers

- Polynomials classifiers do not use a statistical model (make no assumptions) about the distribution of features (and the associated classes) in feature space.
- Their goal is to directly estimate an approximation to the ideal decision function by a polynomial.
- Typically, the designer of the classifier decides what degree of polynomial to use.
- Deriving a polynomial classifier becomes equivalent to computing the coefficients of these polynomials from a labeled training set (supervised training).

Discriminant Function

- Consider a two class problem, of the form a feature vector $\vec{c}$ either belongs to a class or not.
- Examples:
  - car/non-car
  - person/non-person
  - pass quality control/does not pass quality control.
- A discriminant function for class Ω_κ is a polynomial that evaluates to 1 if the feature vector $\vec{c}$ belongs to that class. Otherwise it evaluates to zero.

$$d_\kappa(\vec{c}) = \begin{cases} 1 & \text{if } \vec{c} \in \Omega_\kappa \\ 0 & \text{otherwise} \end{cases}$$

Assumption 1

1. Classification is done by using *K* (*K*= number of classes) discriminant functions (Trennfunktionen).

$$d_1(\vec{c}), d_2(\vec{c}), \ldots, d_K(\vec{c})$$

- We have as many discriminant functions as classes.
- Where in the statistical classifiers we had as many posterior probabilities as we had classes, we now have discriminant functions.
- We decide for the class Ω_λ that achieves the maximum discrimination/separation.

$$\lambda = \underset{\kappa}{\operatorname{argmax}}\, d_\kappa(\vec{c})$$

Assumption 2

2. We assume that these *K* discriminant functions, $d_\kappa(\vec{c})$, belong to a parametric family of functions:

$$d_\kappa(\vec{c}) \in d(\vec{c}, \vec{a}_\kappa)$$

where $\vec{a}_\kappa$ are the coefficients of the polynomial $d_\kappa(\vec{c})$.

- For example, if I have parabolas as discriminant functions, the functions are of the form:

$$d_\kappa(\vec{c}) = a\vec{c}^{\,2} + b\vec{c} + e \quad \text{and} \quad \vec{a}_\kappa = (a, b, e)$$

- Instead of a parametric family of pdfs as in the Gaussian classifier, we have a parametric family of functions.

Optimal Decision Function

- Ideally, an optimal decision function should map a vector $\vec{c}$ to class Ω_κ if it truly belongs to Ω_κ:

$$\delta(\vec{c}) = \begin{cases} 1 & \text{if } \vec{c} \in \Omega_\kappa \\ 0 & \text{for all other classes} \end{cases}$$

- Since we have a binary decision function, we can build a binary *K*-dimensional decision vector with 0s for all the wrong classes and 1 only in the correct class Ω_κ.

$$\vec{\delta}(\vec{c}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Linear Discriminant Function

- Key question: How do we estimate the parameters of the discriminant function?
- Consider a linear discriminant function:

$$d_\lambda(\vec{c}) = (a_{\lambda,0}, a_{\lambda,1}, \ldots, a_{\lambda,M}) \begin{bmatrix} 1 \\ c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix}$$

$$d_\lambda(\vec{c}) = \vec{a}_\lambda \vec{c}\,'^T$$

where *M* is the dimensionality of the feature vector $\vec{c} = (c_1, c_2, \ldots, c_M)$, $\vec{c}\,' = (1, c_1, c_2, \ldots, c_M)$ and *M*+1 is the number of coefficients.

- We want to derive the values of $a_{\lambda,i}$ for $i = 0, \ldots, M$ and $\lambda = 1, \ldots, K$ from the training set.

Training Set

- We have a training set *T* composed of *N* pairs of feature vectors and their assigned class:

$$T = \left\{ \left(\vec{c}_l, \Omega_{\kappa(l)} \right), l = 1,2,\ldots,N \right\}$$

  where $\Omega_{\kappa(l)}$ is the class of feature vector $\vec{c}_l$.
- How can we use this training set?
- An ideal discriminant function $d_\lambda(\vec{c})$ would assign a sample $\vec{c}_l$ to its correct class $\Omega_{\kappa(l)}$.
- In other words, if $\lambda = \kappa(l)$ then in an ideal separating function $d_\lambda(\vec{c}) = 1$, while for $\lambda \neq \kappa(l)$ one should get $d_\lambda(\vec{c}) = 0$.

Multiple Equations

- So given a feature vector $\vec{c}_l$, which belongs to some class Ω_λ we can write *K* equations, out of which one will be equal to 1 and for the rest it will be zero.

$$
\begin{aligned}
d_1(\vec{c}_l) &= \vec{a}_\lambda \vec{c}_l'^T = 0 \\
d_2(\vec{c}_l) &= \vec{a}_\lambda \vec{c}_l'^T = 0 \\
&\vdots \\
d_\lambda(\vec{c}_l) &= \vec{a}_\lambda \vec{c}_l'^T = 1 \\
&\vdots \\
d_K(\vec{c}_l) &= \vec{a}_\lambda \vec{c}_l'^T = 0
\end{aligned}
$$

Ideal Discriminant Function

- As previously stated an ideal discriminant function should lead to correct classification decision.
- So the ideal separating function is:

$$d_\lambda(\vec{c}) = \begin{cases} 1 & \text{if } \lambda = \kappa(\mathrm{l}) \\ 0 & \text{if } \lambda \neq \kappa(\mathrm{l}) \end{cases}$$

- In practice, we can not expect to get exactly zero and exactly 1, so we use the following approximations:

$$d_\lambda(\vec{c}) = \begin{cases} \left(d_\lambda(\vec{c}) - 1\right)^2 = \min & \text{if } \lambda = \kappa(\mathrm{l}) \\ \left(d_\lambda(\vec{c})\right)^2 = \min & \text{if } \lambda \neq \kappa(\mathrm{l}) \end{cases}$$

Ideal Discriminant Function – cont.

- We want our polynomial separating functions to approximate as closely as possible the ideal decision function.
- The ideal decision function is $\delta_\kappa(\;)$ and the linear separating function is $d_\kappa(\;)$, where $\vec{\delta} = (\delta_1, \delta_2, \ldots, \delta_\kappa, \ldots, \delta_K)$
- So when computing the discriminant functions, the error we want to minimize is:

$$\varepsilon = \sum_{\kappa=1}^{K} \sum_{l=1}^{N} \left(\delta_\kappa(\vec{c}_l) - d_\kappa(\vec{c}_l) \right)^2$$

Ideal Separating Function – cont.

- The goal of a polynomial classifier is then to derive the polynomial coefficients that minimize the deviation from the ideal decision function:

$$\varepsilon = \sum_{\kappa=1}^{K} \sum_{l=1}^{N} \left(\delta_\kappa(\vec{c}_l) - d_\kappa(\vec{c}_l) \right)^2$$

- In other words find $\vec{a}_\lambda$ such that:

$$\vec{a}_\lambda = \underset{\vec{a}_k}{\arg\min}\, \varepsilon$$

Minimizing ε

- For each $a_{\lambda,i},\ i=0,1,\ldots,M$ we do the following.

$$\frac{\partial \varepsilon}{\partial a_{\lambda,i}} = 0 \Rightarrow \left. \partial \sum_{l=1}^{N} \sum_{\kappa=1}^{K} \left(\delta_\kappa(\vec{c}_l) - d_\kappa(\vec{c}_l)\right)^2 \middle/ \partial a_{\lambda,i} \right. = 0$$

$$\frac{\partial \sum_{l=1}^{N} \sum_{\kappa=1}^{K} \left(\delta_\kappa(\vec{c}_l) - (a_{\kappa,0}, a_{\kappa,1}, \ldots, a_{\kappa,M}) \begin{bmatrix} 1 \\ c_{l,1} \\ c_{l,2} \\ \vdots \\ c_{l,M} \end{bmatrix} \right)^2}{\partial a_{\lambda,i}} = 0$$

Minimizing ε - cont

$$-2\sum_{l=1}^{N}\sum_{\kappa=1}^{K}\left(\delta_\kappa(\vec{c}_l)-(a_{\kappa,0},a_{\kappa,1},\ldots,a_{\kappa,M})\begin{bmatrix}1\\c_{l,1}\\c_{l,2}\\\vdots\\c_{l,M}\end{bmatrix}\right)c_{l,i}=0$$

$$-2\sum_{l=1}^{N}\sum_{\kappa=1}^{K}\left(\delta_\kappa(\vec{c}_l)-\vec{a}_\kappa\vec{c}_l^{\,\prime T}\right)c_{l,i}=0$$

- Note that this equation is linear in $(a_{\kappa,0},a_{\kappa,1},\ldots,a_{\kappa,M})$

Solving the Minimization

- We need to repeat this process for each $a_{\lambda,i},\ i = 0,1,\ldots,M$
- We get a system of linear equations:

$$\left.\begin{aligned}
-2\sum_{l=1}^{N}\sum_{\kappa=1}^{K}\left(\delta_\kappa(\vec{c}_l) - \vec{a}_\kappa \vec{c}_l^{\,\prime T}\right) &= 0 \\
-2\sum_{l=1}^{N}\sum_{\kappa=1}^{K}\left(\delta_\kappa(\vec{c}_l) - \vec{a}_\kappa \vec{c}_l^{\,\prime T}\right)c_{l,1} &= 0 \\
-2\sum_{l=1}^{N}\sum_{\kappa=1}^{K}\left(\delta_\kappa(\vec{c}_l) - \vec{a}_\kappa \vec{c}_l^{\,\prime T}\right)c_{l,2} &= 0 \\
&\vdots \\
-2\sum_{l=1}^{N}\sum_{\kappa=1}^{K}\left(\delta_\kappa(\vec{c}_l) - \vec{a}_\kappa \vec{c}_l^{\,\prime T}\right)c_{l,M} &= 0
\end{aligned}\right\}
Q\begin{bmatrix} a_{\kappa,0} \\ a_{\kappa,1} \\ a_{\kappa,2} \\ \vdots \\ a_{\kappa,M} \end{bmatrix} = \vec{b} \Rightarrow \begin{bmatrix} a_{\kappa,0} \\ a_{\kappa,1} \\ a_{\kappa,2} \\ \vdots \\ a_{\kappa,M} \end{bmatrix} = Q^{+}\vec{b}$$

Linear Classifier and Gaussian Classifier

- Recall that a linear classifier is equivalent to a Gaussian classifier where the covariance matrix is independent of the class Ω_κ.
- Given a classification problem, one can test quickly how well a linear classifier works. If we get good results, then we most probably have normally distributed features with same covariances in all classes.
- We can then choose to explicitly use a Gaussian classifier, or otherwise exploit the normal distribution of the features.
- A similar process can be applied for quadratic separating functions and normally distributed features with distinct covariances among the different classes.

Higher Order Polynomials

- In higher order polynomials we take powers of the components of the feature vector $\vec{c}$.
- The general form of higher order polynomials is:

$$d_\lambda(\vec{c}) = \sum_{\substack{n=0 \\ l_1+l_2+l_3+\ldots+l_M=n}}^{P} a_{\lambda,n} c_1^{l_1} c_2^{l_2} \cdots c_M^{l_M}$$

where *P* is the degree of the polynomial

- For example, for *P*=2

$$d_\lambda(\vec{c}) = a_{\lambda,0} + a_{\lambda,1}c_1 + a_{\lambda,1}c_2 + \ldots + a_{\lambda,1}c_M +$$
$$+ a_{\lambda,2}c_1^2 + a_{\lambda,2}c_2^2 \ldots + a_{\lambda,2}c_M^2 + a_{\lambda,2}c_1c_2 + a_{\lambda,2}c_1c_3 + \cdots$$

Estimation of the Coefficients

- Note that in the higher order polynomials, the discriminant functions are still linear in the $a_{\lambda,i}$s but not in the components of the vector $\vec{c}$.
- This means that estimation of the coefficients $a_{\lambda,i}$ can be done as before.
- We want to get as closely as possible to the ideal decision function, so we use a similar error function.
- To minimize it we take the partial derivative for each $a_{\lambda,i}$.
- We have a system of equations from our training data which we could solve via SVD.

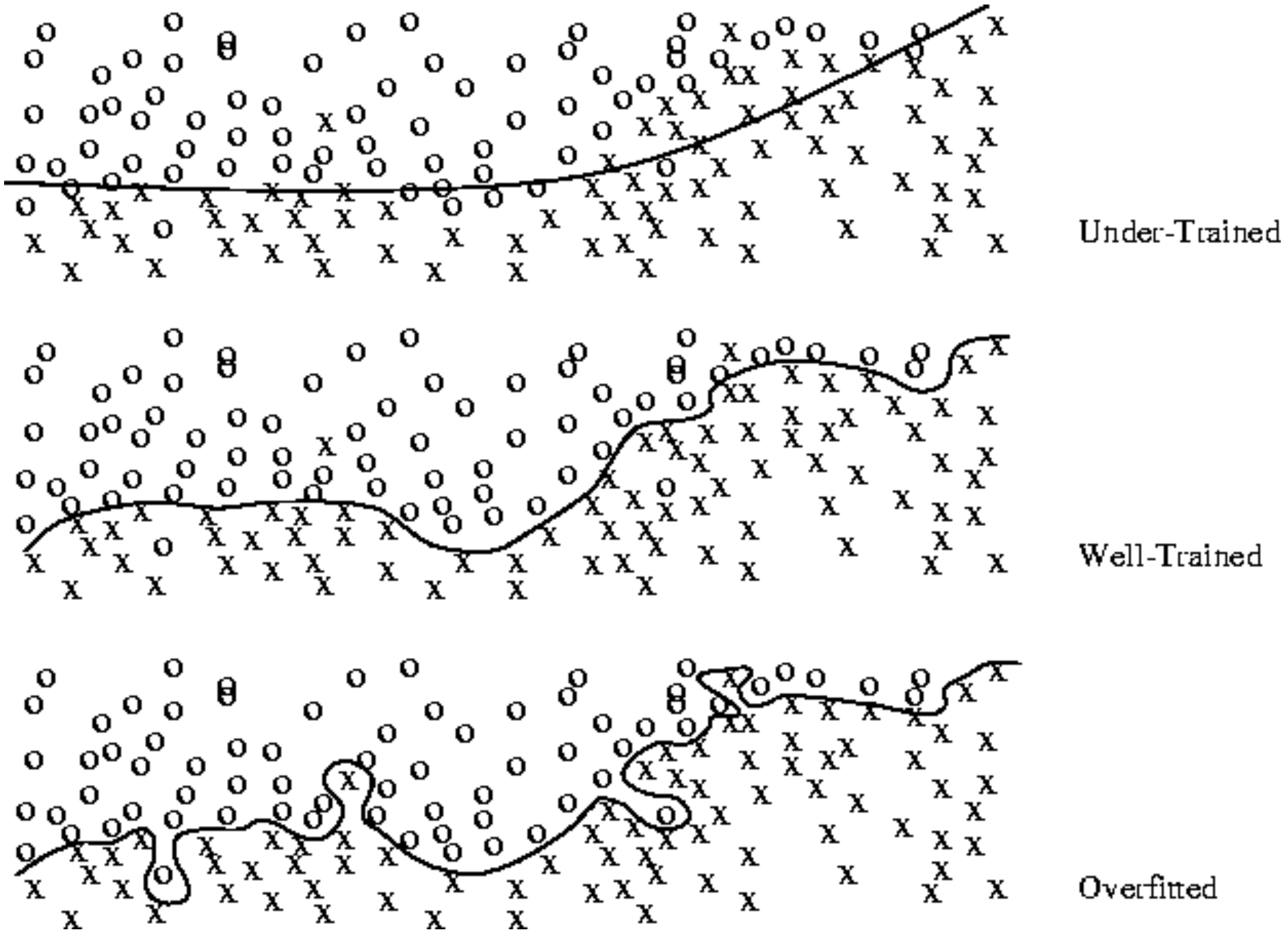
Remarks

- When designing a polynomial classifier one needs:

1. A labeled training set
2. Decide on the degree of the polynomial

- Be careful: from polynomial approximation we know that high order polynomials can perfectly fit the training data, but it may lead to an overfitting problem.
- Data Overfitting: The classifier (or more generally the model) responds to very specific attributes of the data (even noise) that do not generalize to the overall population.

Overfitting Example



Plot courtesy of A. Schmidt http://www.teco.edu/~albrecht/neuro/html/node10.html

More Remarks

- Training is equivalent more or less to solving linear equations.
- If we do not restrict $d_\lambda(\vec{c})$ to a parametric family of functions, and we use a (0,1) cost function with no rejection class, then we will end up with.

$$d_\lambda(\vec{c}) = p(\Omega_\lambda | \vec{c})$$

- In general, because of the so-called Weierstrass principle, polynomial classifiers are considered universal approximations to the Bayesian classifier.