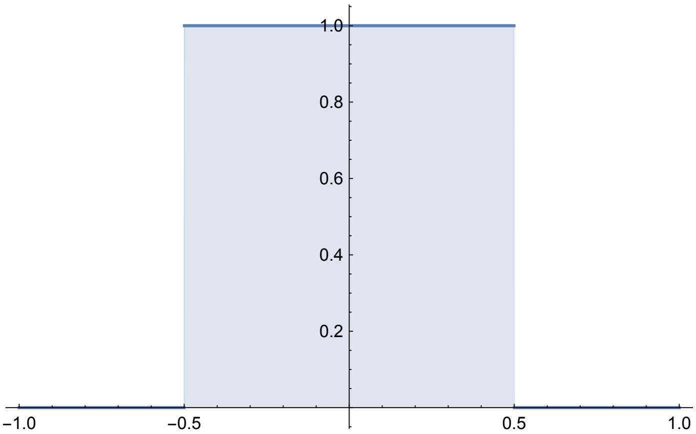
Illustrating the Gibbs-Wilbraham phenomenon

Defining a function on the interval [-1,1)

In[1]:=
```
f[x_] := Piecewise[{{1, Abs[x] ≤ 1/2}}]
```

In[2]:=
```
Plot[f[x], {x, -1, 1}, Filling → Axis]
```

Out[2]=

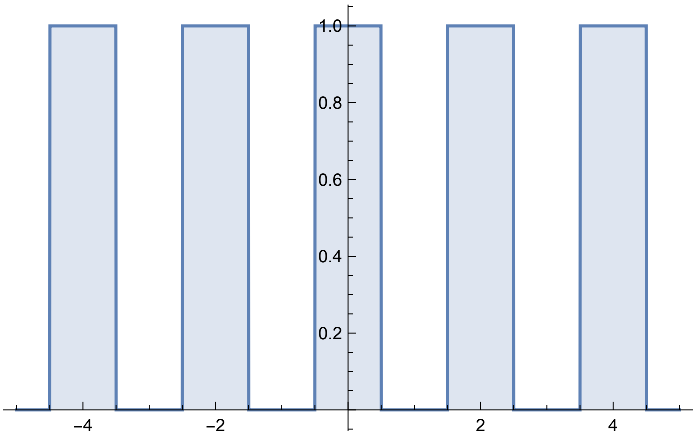


Defining the 2-periodic extension (periodization)

In[3]:=
```
ff[x_] := If[EvenQ[Round[x]], 1, 0]
```

In[4]:=
```
Plot[ff[x], {x, -5, 5}, Filling → Axis]
```

Out[4]=



Computing the Fourier coefficients

In[5]:= codd[n_] := $\int_{-1}^{1} f(t)\, \exp\left(-\pi\, i\, (2n-1)\, t\right)\, dt$

In[6]:=
```
codd[n]
```

Out[6]= $-\dfrac{2\,\mathrm{Cos}[n\,\pi]}{(-1+2\,n)\,\pi}$

In[7]:= ceven[n_] := $\int_{-1}^{1} f(t)\, \exp\left(-\pi\, i\, (2n)\, t\right)\, dt$

In[8]:=
```
ceven[n]
```

Out[8]= $\dfrac{\mathrm{Sin}[n\,\pi]}{n\,\pi}$

Even indexed Fourier coefficients vanish, except for n=0

In[9]:=
```
ceven[0]
```

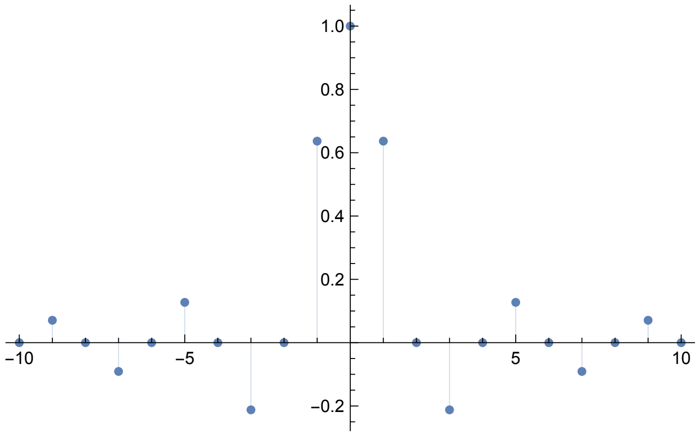
Out[9]= 1

Plotting the Fourier coefficients

In[10]:=
```
c[n_] := If[EvenQ[n], ceven[n/2], codd[(n+1)/2]]
```

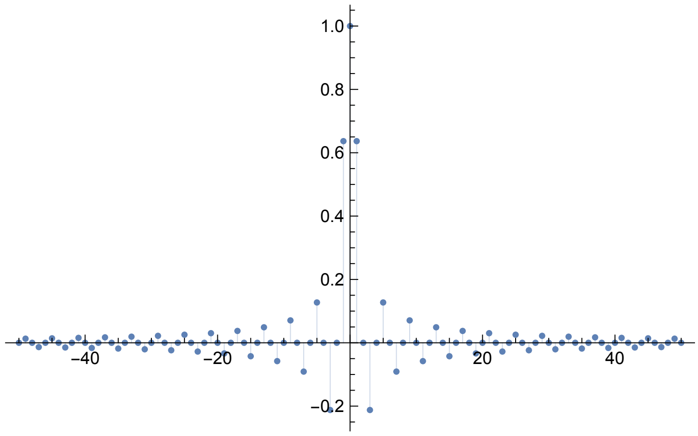
In[11]:=
```
ListPlot[Table[{k, c[k]}, {k, -10, 10}], Filling → Axis, PlotRange → All]
```

Out[11]=



In[12]:=
```
ListPlot[Table[{k, c[k]}, {k, -50, 50}], Filling → Axis, PlotRange → All]
```

Out[12]=



Approximation by the first N+1 terms of the Fourier series

In[13]:= $S[N\_, s\_] := \frac{1}{2} + \sum_{k=1}^{N} \text{codd}(k) \cos(\pi (2k-1) s)$

In[14]:=
```
s2 = S[2, s]
```

Out[14]= $\frac{1}{2} + \frac{2 \text{Cos}[\pi s]}{\pi} - \frac{2 \text{Cos}[3 \pi s]}{3 \pi}$

In[15]:=
```
s3 = S[3, s]
```

Out[15]= $\frac{1}{2} + \frac{2 \text{Cos}[\pi s]}{\pi} - \frac{2 \text{Cos}[3 \pi s]}{3 \pi} + \frac{2 \text{Cos}[5 \pi s]}{5 \pi}$

In[16]:=
```
s5 = S[5, s]
```

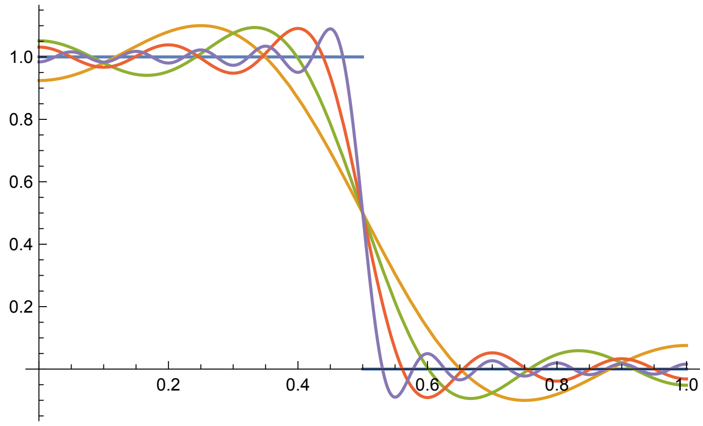
Out[16]= $\frac{1}{2} + \frac{2 \text{Cos}[\pi s]}{\pi} - \frac{2 \text{Cos}[3 \pi s]}{3 \pi} + \frac{2 \text{Cos}[5 \pi s]}{5 \pi} - \frac{2 \text{Cos}[7 \pi s]}{7 \pi} + \frac{2 \text{Cos}[9 \pi s]}{9 \pi}$

In[17]:= `s10 = S[10, s]`

Out[17]=
$$\frac{1}{2}+\frac{2\,\text{Cos}[\pi\,s]}{\pi}-\frac{2\,\text{Cos}[3\,\pi\,s]}{3\,\pi}+\frac{2\,\text{Cos}[5\,\pi\,s]}{5\,\pi}-\frac{2\,\text{Cos}[7\,\pi\,s]}{7\,\pi}+\frac{2\,\text{Cos}[9\,\pi\,s]}{9\,\pi}-\frac{2\,\text{Cos}[11\,\pi\,s]}{11\,\pi}+\frac{2\,\text{Cos}[13\,\pi\,s]}{13\,\pi}-\frac{2\,\text{Cos}[15\,\pi\,s]}{15\,\pi}+\frac{2\,\text{Cos}[17\,\pi\,s]}{17\,\pi}-\frac{2\,\text{Cos}[19\,\pi\,s]}{19\,\pi}$$

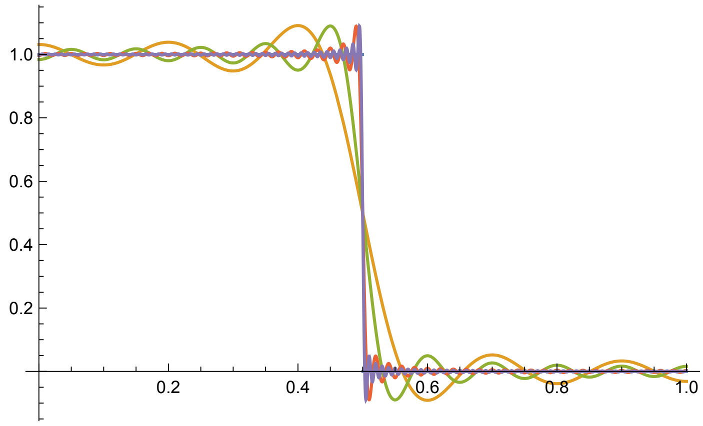
Plotting the truncated series

In[18]:= `Plot[{f[s], s2, s3, s5, s10}, {s, 0, 1}]`

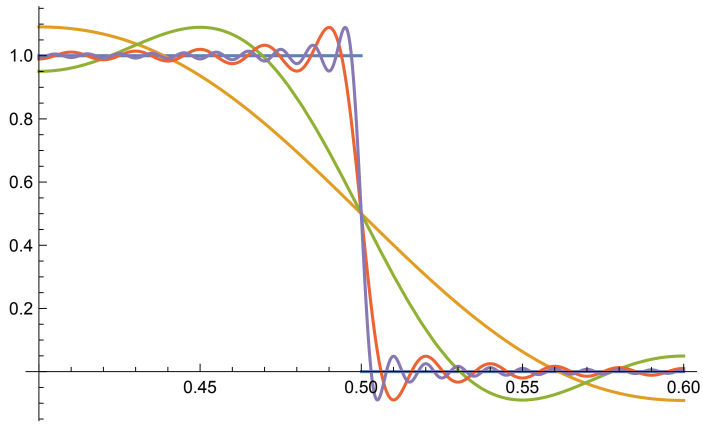
Out[18]=


In[19]:= `s50 = S[50, s]; s100 = S[100, s];`

In[20]:= `Plot[{f[s], s5, s10, s50, s100}, {s, 0, 1}]`

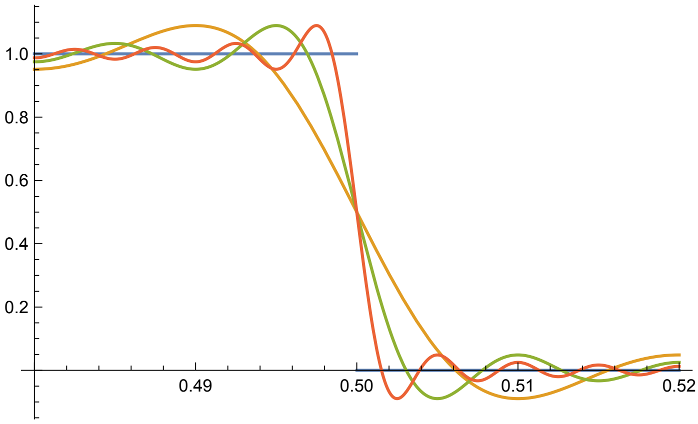
Out[20]=


In[21]:= `Plot[{f[s], s5, s10, s50, s100}, {s, 0.4, 0.6}]`

Out[21]=


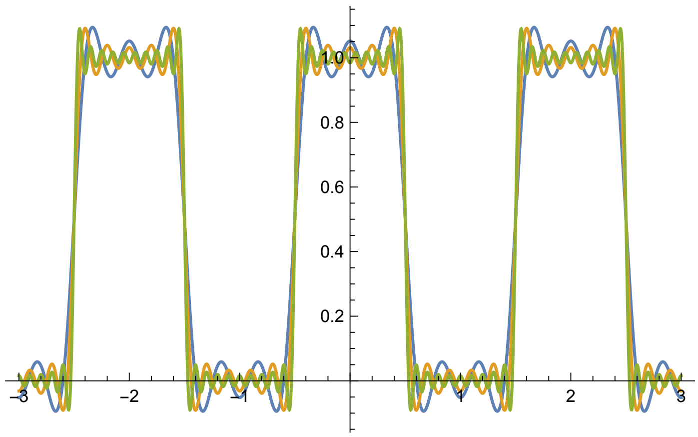
In[22]:=
```
s200 = S[200, s];
```

In[23]:=
```
Plot[{f[s], s50, s100, s200}, {s, 0.48, 0.52}]
```

Out[23]=


The approximations as 2-periodic functions

In[24]:=
```
Plot[{s3, s5, s10}, {s, -3, 3}]
```

Out[24]=


Convergence in the L^2-Norm

In[25]:=
```
d[N_] := NIntegrate[Abs[f[x] - S[N, x]]^2, {x, -1, 1}]
```

In[26]:=
```
{d[3], d[5], d[10], d[20], d[50], d[100]}
```

Out[26]=
```
{0.0334722, 0.0201976, 0.0101237, 0.005065, 0.00202636, 0.00101318}
```

In[27]:=
```
ListPlot[Table[d[n], {n, 1, 20}], Filling → Axis, PlotRange → All]
```

Out[27]=