Biorthogonal Filter Pairs und Wavelets

WTBV

January 20, 2016

- Up to now: *orthogonal* wavelet transforms with filters of finite length $L+1$, based on pairs of filters

$$\begin{aligned} &\text{low-pass filter} \qquad \boldsymbol{h} = (h_0, h_1, \ldots, h_L) \\ &\text{high-pass filter} \qquad \boldsymbol{g} = (g_0, g_1, \ldots, g_L) \end{aligned}$$

  defining an orthogonal transform of signals (of finite length)
- written in matrix form as

$$W_N = \begin{bmatrix} H_N \\ G_N \end{bmatrix} \quad \text{with} \quad W_N^{-1} = W_N^{\dagger}$$

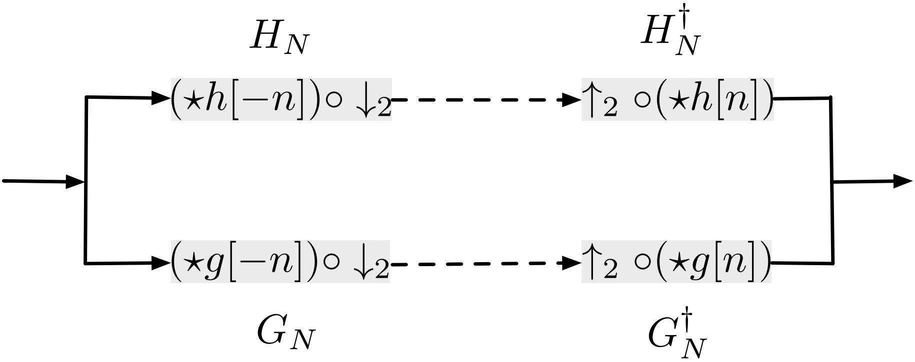


Figure: Filter bank scheme of orthogonal WT

- Orthogonality as specified by

$$I_N = W_N \, W_N^\dagger \quad \text{resp.} \quad I_N = W_N^\dagger \, W_N$$

is equivalent to three idenities

$$\begin{aligned} G_N \, G_N^\dagger = I_{N/2} = H_N \, H_N^\dagger \\ G_N \, H_N^\dagger = 0_{N/2} = H_N \, G_N^\dagger \\ I_N = G_N^\dagger \, G_N + H_N^\dagger \, H_N \end{aligned}$$

- The third identity expresses the *reconstruction* property

- Looking at the frequency picture:

$$\begin{aligned} |H(\omega)|^2 + |H(\omega+\pi)|^2 &= 2 \\ |G(\omega)|^2 + |G(\omega+\pi)|^2 &= 2 \\ H(\omega)\,\overline{G(\omega)} + H(\omega+\pi)\,\overline{G(\omega+\pi)} &= 0 \\ H(0) = G(\pi) &= \sqrt{2} \\ H(\pi) = G(0) &= 0 \end{aligned}$$

- the last two equation expressing low-pass properties of $\boldsymbol{h}$, resp. the high-pass properties of $\boldsymbol{g}$

- The reason to deviate from this standard scheme comes from the following observations:
  - *Symmetric* filters (and wavelets) often give visually better reconstruction results (e.g. when using wavelets for image compression)
  - Apart from the HAAR-filter there are no other symmetric scaling filters from which an orthogonal transform scheme (as above) can be built

- The idea to be able to use symmetric filters leads to a more general approach:
  - Take two pairs of filters
    - one pair $(\boldsymbol{h}, \widetilde{\boldsymbol{h}})$ of low-pass filters
    - one pair $(\boldsymbol{g}, \widetilde{\boldsymbol{g}})$ of high-pass filters
  - length and index ranges of these filters are not yet specified – but the filters shall have finite length
  - it is not required that $\boldsymbol{h}$ and $\boldsymbol{g}$ have the same length
- This leads to the so-called *bi-orthogonal* set-up

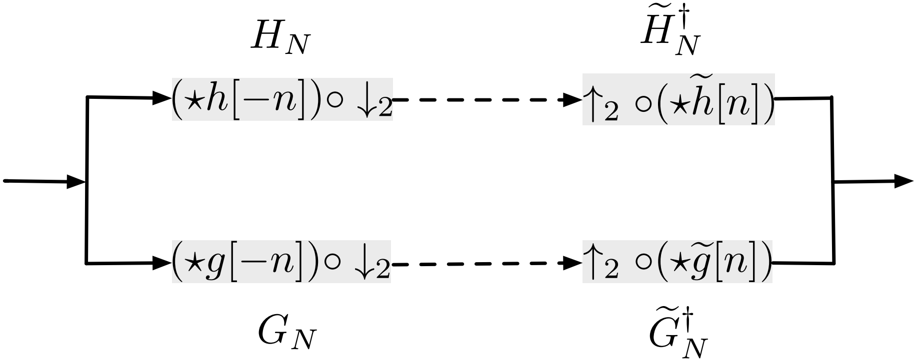


Figure: Filter bank scheme of a bi-orthogonal WT

- The transformation matrices for analysis and synthesis are given by

$$\text{analysis:} \quad W_N = \begin{bmatrix} H_N \\ G_N \end{bmatrix} \qquad \text{synthesis:} \quad \widetilde{W}_N = \begin{bmatrix} \widetilde{H}_N \\ \widetilde{G}_N \end{bmatrix}$$

and these matrices are required to be inverse to each other:

$$W_N^{-1} = \widetilde{W}_N^{\dagger}$$

- which means

$$W_N \, \widetilde{W}_N^{\dagger} = \widetilde{W}_N^{\dagger} \, W_N = I_N$$

- and in more detail

$$\begin{aligned} G_N \, \widetilde{G}_N^{\dagger} &= I_{N/2} = H_N \, \widetilde{H}_N^{\dagger} \\ G_N \, \widetilde{H}_N^{\dagger} &= 0_{N/2} = H_N \, \widetilde{G}_N^{\dagger} \\ I_N &= \widetilde{G}_N^{\dagger} \, G_N + \widetilde{H}_N^{\dagger} \, H_N \end{aligned}$$

- The different ways to express these requirements

transformation matrices $\leftrightarrow$ filter coefficients $\leftrightarrow$ frequency representation

$$(H_N, G_N) \qquad (\boldsymbol{h}, \boldsymbol{g}) \qquad (H(\omega), G(\omega))$$
$$(\widetilde{H}_N, \widetilde{G}_N) \qquad (\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{g}}) \qquad (\widetilde{H}(\omega), \widetilde{G}(\omega))$$

$$H_N \widetilde{H}_N^\dagger = I_{N/2} \Leftrightarrow \sum_k \widetilde{h}_k h_{k-2m} = \delta_{m,0}$$
$$\Leftrightarrow \widetilde{H}(\omega)\overline{H}(\omega) + \widetilde{H}(\omega+\pi)\overline{H}(\omega+\pi) = 2 \quad (1)$$

$$G_N \widetilde{G}_N^\dagger = I_{N/2} \Leftrightarrow \sum_k \widetilde{g}_k g_{k-2m} = \delta_{m,0}$$
$$\Leftrightarrow \widetilde{G}(\omega)\overline{G}(\omega) + \widetilde{G}(\omega+\pi)\overline{G}(\omega+\pi) = 2 \quad (2)$$

$$H_N \widetilde{G}_N^\dagger = 0_{N/2} \Leftrightarrow \sum_k \widetilde{g}_k h_{k-2m} = 0$$
$$\Leftrightarrow \widetilde{H}(\omega)\overline{G}(\omega) + \widetilde{H}(\omega+\pi)\overline{G}(\omega+\pi) = 0 \quad (3)$$

$$G_N \widetilde{H}_N^\dagger = 0_{N/2} \Leftrightarrow \sum_k \widetilde{h}_k g_{k-2m} = 0$$
$$\Leftrightarrow \widetilde{G}(\omega)\overline{H}(\omega) + \widetilde{G}(\omega+\pi)\overline{H}(\omega+\pi) = 0 \quad (4)$$

Definition

A pair $(\boldsymbol{h}, \widetilde{\boldsymbol{h}})$ *of (low-pass) filters of finite length is said to be a biorthogonal filter pair if condition* (1) *holds*

$$\widetilde{H}(\omega)\overline{H}(\omega) + \widetilde{H}(\omega + \pi)\overline{H}(\omega + \pi) = 2 \quad (1)$$

Proposition

If $(\boldsymbol{h}, \widetilde{\boldsymbol{h}})$ *is a biorthogonal filter pair, i.e.,* (1) *holds,*
and if one defines a filter pair $(\boldsymbol{g}, \widetilde{\boldsymbol{g}})$ *by setting*

$$G(\omega) = e^{i(n\omega+b)}\overline{\widetilde{H}(\omega + \pi)} \qquad \widetilde{G}(\omega) = e^{i(n\omega+b)}\overline{H(\omega + \pi)}$$

with odd $n \in \mathbb{Z}$ *and* $b \in \mathbb{R}$*, the conditions* (2), (3) *und* (4) *and reconstructibility are automatically satisfied*

- For the filter coefficients these setting give

$$g_k = -e^{ib}(-1)^k \widetilde{h}_{n-k}, \quad \widetilde{g}_k = -e^{ib}(-1)^k h_{n-k}.$$

- One usually puts $b = \pi$ (in order to have real filter coefficients!) and $n = 1$, so that

$$g_k = (-1)^k \widetilde{h}_{1-k}, \quad \widetilde{g}_k = (-1)^k h_{1-k}$$

- Note: filter $\boldsymbol{h}$ determines filter $\widetilde{\boldsymbol{g}}$ – in particular: they have the same length – and similarly filter $\widetilde{\boldsymbol{h}}$ determines the filter $\boldsymbol{g}$

  Filters $\boldsymbol{h}$ and $\widetilde{\boldsymbol{h}}$ do not need to have the same length, but their choice is not completely arbitrary – see the following proposition

- Example

$$\boldsymbol{h} = \frac{\sqrt{2}}{4}(-2, 4, 3, -2, 1) = (h_{-2}, \dots, h_2)$$
$$\widetilde{\boldsymbol{h}} = \frac{\sqrt{2}}{4}(1, 2, 1) = (\widetilde{h}_{-1}, \widetilde{h}_0, \widetilde{h}_1)$$

- frequency representation

$$H(\omega) = \frac{\sqrt{2}}{4}(-2e^{-2i\omega} + \cdots + 1e^{2i\omega})$$
$$\widetilde{H}(\omega) = \frac{\sqrt{2}}{4}(e^{-i\omega} + 2 + e^{i\omega})$$

- check that

$$H(0) = \widetilde{H}(0) = \sqrt{2}$$
$$H(\pi) = \widetilde{H}(\pi) = 0$$
$$\widetilde{H}(\omega)\overline{H(\omega)} = \frac{1}{8}(e^{-3i\omega} + 8 + 9e^{i\omega} - 2e^{3i\omega})$$
$$\widetilde{H}(\omega+\pi)\overline{H(\omega+\pi)} = \frac{1}{8}(-e^{-3i\omega} + 8 - 9e^{i\omega} + 2e^{3i\omega})$$

- ... which gives

$$\widetilde{H}(\omega)\overline{H(\omega)} + \widetilde{H}(\omega+\pi)\overline{H(\omega+\pi)} = 2$$

  so that the necessary requirement (1) is satisfied

- As for the filters $\boldsymbol{g}$ and $\widetilde{\boldsymbol{g}}$:

$$\boldsymbol{g} = \frac{\sqrt{2}}{4}(1,-2,1) = (g_0, g_1, g_2)$$

$$\widetilde{\boldsymbol{g}} = \frac{\sqrt{2}}{4}(-1,-2,-3,4,2) = (\widetilde{g}_{-1}, \ldots, \widetilde{g}_3)$$

Transformation matrices for signals of length 8:

- analysis transform

$$W_8 = \begin{bmatrix} H_8 \\ G_8 \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & h_2 & 0 & 0 & 0 & h_{-2} & h_{-1} \\ h_{-2} & h_{-1} & h_0 & h_1 & h_2 & 0 & 0 & 0 \\ 0 & 0 & h_{-2} & h_{-1} & h_0 & h_1 & h_2 & 0 \\ h_2 & 0 & 0 & 0 & h_{-2} & h_{-1} & h_0 & h_1 \\ g_0 & g_1 & g_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_0 & g_1 & g_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_0 & g_1 & g_2 & 0 \\ g_2 & 0 & 0 & 0 & 0 & 0 & g_0 & g_1 \end{bmatrix}$$

- synthesis transform

$$\widetilde{W}_8 = \begin{bmatrix} \widetilde{H}_8 \\ \widetilde{G}_8 \end{bmatrix} = \begin{bmatrix} \widetilde{h}_0 & \widetilde{h}_1 & 0 & 0 & 0 & 0 & 0 & \widetilde{h}_{-1} \\ 0 & \widetilde{h}_{-1} & \widetilde{h}_0 & \widetilde{h}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{h}_{-1} & \widetilde{h}_0 & \widetilde{h}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \widetilde{h}_{-1} & \widetilde{h}_0 & \widetilde{h}_1 \\ \widetilde{g}_0 & \widetilde{g}_1 & \widetilde{g}_2 & \widetilde{g}_3 & 0 & 0 & 0 & \widetilde{g}_{-1} \\ 0 & \widetilde{g}_{-1} & \widetilde{g}_0 & \widetilde{g}_1 & \widetilde{g}_2 & \widetilde{g}_3 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{g}_{-1} & \widetilde{g}_0 & \widetilde{g}_1 & \widetilde{g}_2 & \widetilde{g}_3 \\ \widetilde{g}_2 & \widetilde{g}_3 & 0 & 0 & 0 & \widetilde{g}_{-1} & \widetilde{g}_0 & \widetilde{g}_1 \end{bmatrix}$$

Proposition

For a biorthogonal filter pair $(\boldsymbol{h}, \widetilde{\boldsymbol{h}})$ *with*
$\boldsymbol{h} = (h_\ell, \ldots, h_L)$ *(i.e., length* $N = L - \ell + 1$*) and*
$\widetilde{\boldsymbol{h}} = (\widetilde{h}_{\widetilde{\ell}}, \ldots, \widetilde{h}_{\widetilde{L}})$*, (i.e., filter length* $\widetilde{N} = \widetilde{L} - \widetilde{\ell} + 1$*)*
the following holds:

1. *The lengths* N *and* $\widetilde{N}$ *have the same parity, i.e.,* $N \equiv \widetilde{N} \mod 2$
2. *If* N *and* $\widetilde{N}$ *are both even, then* $L \equiv \widetilde{L} \mod 2$
3. *If* N *and* $\widetilde{N}$ *are both odd, then* $L \not\equiv \widetilde{L} \mod 2$

Definition

A filter $\boldsymbol{h} = (h_\ell, \ldots, h_L)$ *is said to be symmetric if*

- $h_k = h_{-k}$ $(k \in \mathbb{Z})$*, if* $\ell = -L$ *(odd length), or if*
- $h_k = h_{1-k}$ $(k \in \mathbb{Z})$*, if* $\ell = -L + 1$ *(even length)*

Proposition

If $(\boldsymbol{h}, \widetilde{\boldsymbol{h}})$ *is a biorthogonal filter pair with symmetric filters, where* $L < \widetilde{L}$*, then the orthogonality conditions can be written as*

$$\sum_{k=p}^{L} h_k \, \widetilde{h}_{k-2m} = \delta_{0,m} \quad (0 \leq m \leq L),$$

where $p = -L$ *(if* N *is even) or* $p = -L + 1$ *(if* N *is odd)*

Example: Construction of a symmetric biorthogonal filter pair

- $\boldsymbol{h} = (h_0, h_1)$ a symmetric filter of length 2, so $h_0 = h_1$, and $\widetilde{\boldsymbol{h}} = (\widetilde{h}_{-2}, \widetilde{h}_{-1}, \widetilde{h}_0, \widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3)$ a symmetric filter of length 6, which means $\widetilde{h}_0 = \widetilde{h}_1$, $\widetilde{h}_{-1} = \widetilde{h}_2$, $\widetilde{h}_{-2} = \widetilde{h}_3$
- From the Fourier series

$$H(\omega) = h_0 + h_1 e^{i\omega}, \quad \widetilde{H}(\omega) = \widetilde{h}_{-2} e^{-2i\omega} + \cdots + \widetilde{h}_3 e^{3i\omega}$$

  the low-pass requirements imply conditions to be satisfied by the coefficients:

$$H(0) = 2\,h_0 \overset{!}{=} \sqrt{2} \quad \Rightarrow \quad h_0 = h_1 = \frac{1}{\sqrt{2}}$$

$$H(\pi) \overset{!}{=} 0 \quad \text{holds!}$$

$$\widetilde{H}(0) \overset{!}{=} \sqrt{2} \quad \Rightarrow \quad \widetilde{h}_1 + \widetilde{h}_2 + \widetilde{h}_3 = \frac{1}{\sqrt{2}}$$

$$H(\pi) \overset{!}{=} 0 \quad \Rightarrow \quad \widetilde{h}_3 - \widetilde{h}_2 + \widetilde{h}_1 - \widetilde{h}_1 + \widetilde{h}_2 - \widetilde{h}_3 = 0 \quad \text{holds!}$$

- Now about orthogonality

$$h_0\,\widetilde{h}_0 + h_1\,\widetilde{h}_1 \stackrel{!}{=} 1 \quad\Rightarrow\quad \widetilde{h}_0 = \widetilde{h}_1 = \frac{1}{\sqrt{2}}$$

$$h_0\,\widetilde{h}_{-2} + h_1\,\widetilde{h}_{-1} \stackrel{!}{=} 0 \quad\Rightarrow\quad \widetilde{h}_{-2} = -\widetilde{h}_{-1} = \frac{a}{\sqrt{2}}$$

  with a parameter $a \neq 0$

- This gives

$$\boldsymbol{h} = \frac{1}{\sqrt{2}}(1,1) \qquad \widetilde{\boldsymbol{h}} = \frac{1}{\sqrt{2}}(a,-a,1,1,-a,a) \qquad \square$$

- To construct a biorthogonal filter pair $(\boldsymbol{h}, \widetilde{\boldsymbol{h}})$ of low-pass filters of finite length one can proceed as follows:
  - First choose a symmetric filter $\widetilde{\boldsymbol{h}}$ such that sufficiently many low-pass requirements $\widetilde{H}^{(m)}(\pi) = 0$ $(m = 0, 1, 2, \ldots)$ are satisfied. These are *linear* conditions imposed on the coefficients
  - Choose the length of the filter $\boldsymbol{h}$, where the lengths of $\boldsymbol{h}$ and $\widetilde{\boldsymbol{h}}$ should not differ too much, so that the synthesis filters $\widetilde{\boldsymbol{h}}$ and $\widetilde{\boldsymbol{g}}$ have similar properties w.r.t. smoothness
  - Now try to solve the *linear* system (1) for the coefficients of $\boldsymbol{h}$:

$$\widetilde{H}(\omega)\overline{H}(\omega) + \widetilde{H}(\omega + \pi)\overline{H}(\omega + \pi) = 2$$

  - Observe: Asking for symmetry reduces the number of variables to be determined, but also reduces the chances of solvability!
  - If the linear system turns out not to be solvable, one has to increase the proposed length of the filter $\boldsymbol{h}$
- Note that reconstruction quality (smoothness) increases with filter length

- Symmetric low-pass filters (of odd length)
  - For even N one has

$$\cos^N(\omega/2) = \frac{1}{2^N} \sum_{k=-N/2}^{N/2} \binom{N}{N/2+k} e^{ik\omega}$$

  - Hence

$$H(\omega) = \sqrt{2}\cos^N(\omega/2)$$

    is the Fourier series of a symmetric low-pass filter $(h_{-N/2}, \ldots, h_{N/2})$ of length $N+1$
  - The coefficients are

$$h_k = \frac{\sqrt{2}}{2^N}\binom{N}{N/2+k} \qquad -N/2 \le k \le N/2$$

$$h_{k-N/2} = \frac{\sqrt{2}}{2^N}\binom{N}{k} \qquad 0 \le k \le N$$

- Symmetric low-passfilter (of even length)
  - For odd N one has

$$e^{i\omega/2}\cos^N(\omega/2) = \frac{1}{2^N}\sum_{k=-(N-1)/2}^{(N+1)/2}\binom{N}{(N-1)/2+k}e^{ik\omega}$$

  - Hence

$$H(\omega) = \sqrt{2}e^{i\omega/2}\cos^N(\omega/2)$$

    is the Fourier series of a symmetric low-pass filter $(h_{-(N-1)/2}, \ldots, h_{(N+1)/2})$ of length $N+1$
  - The coefficients are

$$h_k = \frac{\sqrt{2}}{2^N}\binom{N}{(N-1)/2+k} \quad -(N-1)/2 \le k \le (N+1)/2$$

$$h_{k-(N-1)/2} = \frac{\sqrt{2}}{2^N}\binom{N}{k} \quad 0 \le k \le N$$

- The *spline functions* $B_N(t)$ are defined inductively

$$B_0(t) = \chi_{[-1/2,1/2)}(t)$$

$$B_{N+1}(t) = B_0(t) \star B_N(t) = \int_{-1/2}^{1/2} B_N(t-s)\,ds$$

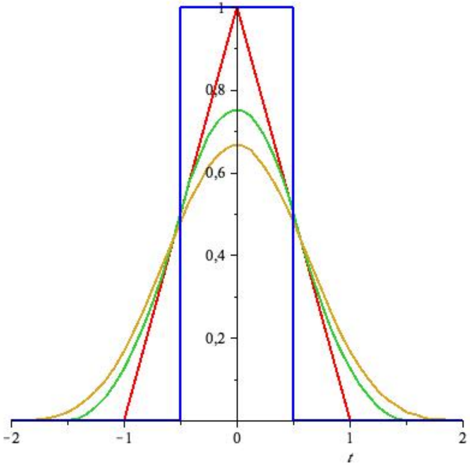
  $B_N(t)$ is the N-fold convolution power of the basis function $B_0(t)$

- An important property of these functions: they satisfy a *scaling idenity*:

$$B_N(t) = \sum_{k=0}^{N+1} \frac{1}{2^N} \binom{N+1}{k} B_N(2t + \lceil N/2 \rceil - k + 1)$$

- The scaling coefficients are (up to a constant factor) the filter coefficients of the *spline filters* defined above — which explains the naming

Graphical display of the spline functions $B_0(t), B_1(t), B_2(t), B_3(t)$



- Taking (as above)
  - $H(\omega) = \sqrt{2}\cos^N(\omega/2)$ as the Fourier series of a symmetric spline filter $\boldsymbol{h} = (h_{-\ell}, \ldots, h_\ell)$ of odd length $N+1 = 2\ell+1$ (for even $N = 2\ell$), resp.
  - $H(\omega) = \sqrt{2}e^{i\omega/2}\cos^N(\omega/2)$ as the Fourier series of a symmetric spline filter $\boldsymbol{h} = (h_{-\ell}, \ldots, h_{\ell+1})$ of even length $N+1 = 2\ell+2$ (for odd $N = 2\ell+1$)
  - then orthogonal symmetric filters fitting to this choice can be constructed using the DAUBECHIES polynomials

$$P_M(z) = \sum_{m=0}^{M} \binom{M+m}{m} z^m$$

Definition

Let $\widetilde{N}$ and N have the same parity.

- *If $N = 2\ell$ and $\widetilde{N} = 2\widetilde{\ell}$ are both even, then define a filter $\widetilde{\boldsymbol{h}}$ through its Fourier series*

$$\widetilde{H}(\omega) = \sqrt{2}\cos^{\widetilde{N}}(\omega/2)P_{\ell+\widetilde{\ell}-1}(\sin^2(\omega/2))$$

- *If $N = 2\ell + 1$ and $\widetilde{N} = 2\widetilde{\ell} + 1$ are both odd, then define a filter $\widetilde{\boldsymbol{h}}$ through its Fourier series*

$$\widetilde{H}(\omega) = \sqrt{2}e^{i\omega/2}\cos^{\widetilde{N}}(\omega/2)P_{\ell+\widetilde{\ell}}(\sin^2(\omega/2))$$

Proposition

With the choice of the previous definition, the following holds for the filter $\widetilde{\boldsymbol{h}}$*:*

1. *filter* $\widetilde{\boldsymbol{h}}$ *has length length* $2\widetilde{N} + N - 1$
2. *filter* $\widetilde{\boldsymbol{h}}$ *is symmetric*
3. *filter* $\widetilde{\boldsymbol{h}}$ *is a low-pass filter*
4. *filters* $\boldsymbol{h}$ *and* $\widetilde{\boldsymbol{h}}$ *are orthogonal*

For the *proof* consider the case where N and $\widetilde{N}$ are both even, i.e. $N = 2\ell$, $\widetilde{N} = 2\widetilde{\ell}$. (The odd case can be treates similarly)

- ad 1./2.
  - Write both factors $\cos^{\widetilde{N}}(\omega/2)$ and $P_{\ell+\widetilde{\ell}-1}(\sin^2(\omega/2))$ as series in $e^{i\omega}$, then

$$\cos^{\widetilde{N}}(\omega/2) = \sum_{k=-\widetilde{\ell}}^{\widetilde{\ell}} \alpha_k e^{ik\omega}$$

    where the sequence of coefficients $(\alpha_{-\ell}, \ldots, \alpha_{\ell})$ is symmetric, since the left-hand side is an even function of ω ist
  - Furthermore, for a similar reason,

$$P_{\ell+\widetilde{\ell}-1}(\sin^2(\omega/2)) = \sum_{m=-\ell-\widetilde{\ell}+1}^{\ell+\widetilde{\ell}-1} \beta_m e^{im\omega}$$

    with a symmetric sequence of coefficients $(\beta_{-\ell-\widetilde{\ell}+1}, \ldots, \beta_{\ell+\widetilde{\ell}-1})$

- ad 1./2. (seq.)
  - Therefore the product has the form

$$\cos^{\widetilde{N}}(\omega/2) \cdot P_{\ell+\widetilde{\ell}-1}(\sin^2(\omega/2)) = \sum_{n=-2\widetilde{\ell}-\ell+1}^{2\widetilde{\ell}+\ell-1} \gamma_n e^{in\omega}$$

    with a symmetric sequence of coefficients $(\gamma_{-2\widetilde{\ell}-\ell+1}, \ldots, \gamma_{2\widetilde{\ell}+\ell-1})$, because the convolution of symmetric sequences is again symmetric
  - The length is $2(2\widetilde{\ell} + \ell - 1) + 1 = 2\widetilde{N} + N - 1$

- ad 3.
  - Obviously $\widetilde{H}(0) = \sqrt{2}$ and $\widetilde{H}(\pi) = 0$
- ad 4.
  - Setting $z = e^{i\omega}$ and $y = \sin^2(\omega/2)$ one has

$$\begin{aligned} H(\omega)\widetilde{H}(\omega) &= 2\cos^{N+\widetilde{N}}(\omega/2)\, P_{\ell+\widetilde{\ell}-1}(\sin^2(\omega/2)) \\ &= 2(1-y)^{\ell+\widetilde{\ell}}\, P_{\ell+\widetilde{\ell}-1}(y) \\ &= 2\widehat{P}_{N+\widetilde{N}-1}(z) = 2\widehat{P}_{N+\widetilde{N}-1}(e^{i\omega}) \end{aligned}$$

  - Reminder: an important property of the DAUBECHIES polynomials is

$$\widehat{P}_{2M-1}(z) + \widehat{P}_{2M-1}(-z) = 1$$

  - As desired, one gets

$$\begin{aligned} H(\omega)\widetilde{H}(\omega) &+ H(\omega+\pi)\widetilde{H}(\omega+\pi) \\ &= 2(\widehat{P}_{N+\widetilde{N}-1}(z) + \widehat{P}_{N+\widetilde{N}-1}(-z)) = 2 \end{aligned}$$

  - NB: Complex conjugation does not show up because the filters are real

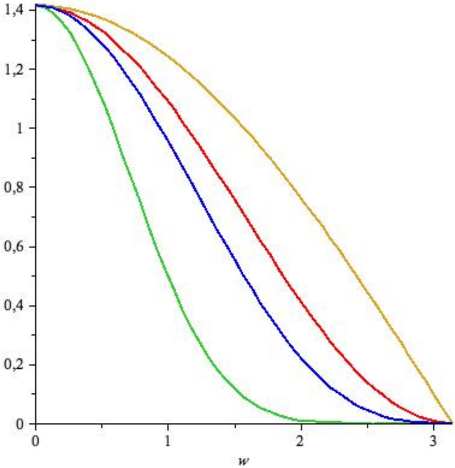


Figure: Frequency representations of the Bspline filters of length 2,3,4,9

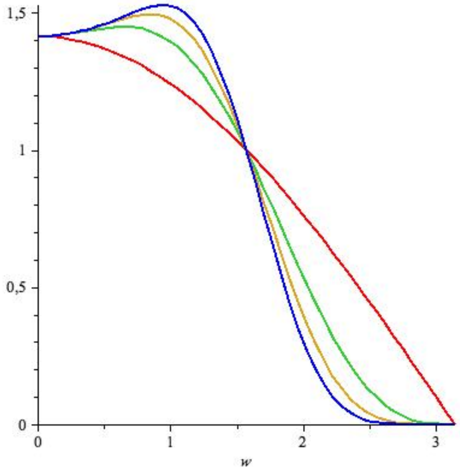


Figure: Bspline filter partners $K_{1,1}, K_{3,1}, K_{5,1}, K_{7,1}$

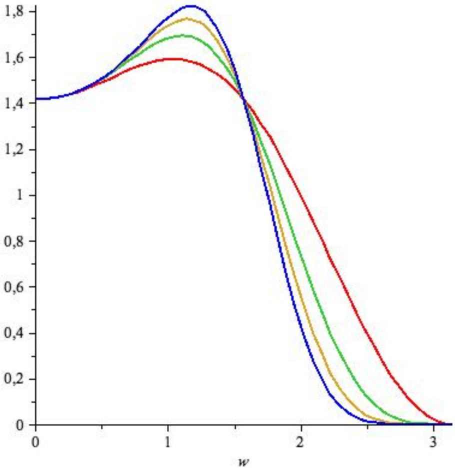


Figure: Bspline filter partners $K_{2,2}, K_{2,4}, K_{2,6}, K_{2,8}$

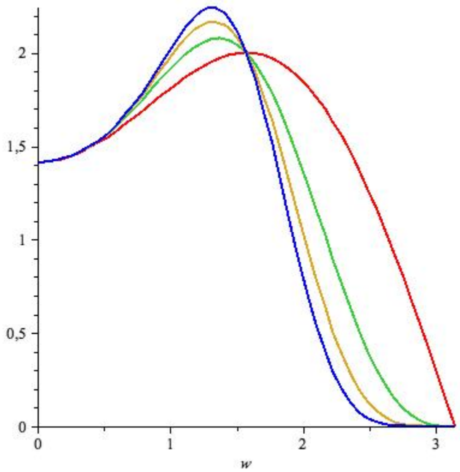


Figure: Bspline filter partners $K_{1,3}, K_{3,3}, K_{5,3}, K_{7,3}$

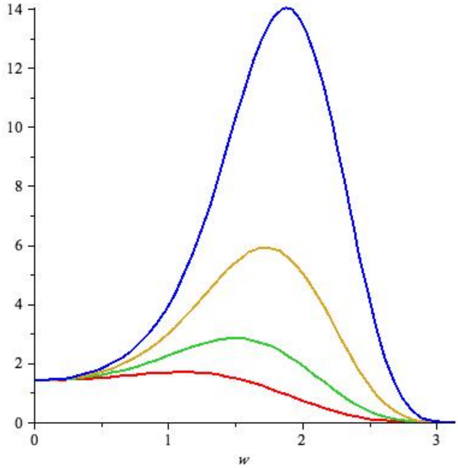


Figure: Bspline filter partners $K_{4,2}, K_{4,4}, K_{4,6}, K_{4,8}$

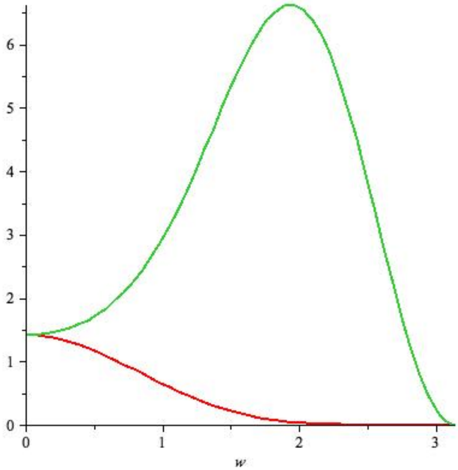


Figure: (7,9) Bspline filter pair

- The DAUBECHIES polynomials

$$P_M(z) = \sum_{m=0}^{M} \binom{M+m}{m} z^m$$

satisfy the fundamental identity

$$(1-z)^{M+1} P_M(z) + z^{M+1} P_M(1-z) = 1$$

- The polynomials $(1-z)^{M+1}$ and z^{M+1} have no common roots (obviously!), hence do not have a proper common divisor.

  Reading the above identity as a Bezout identity for polynomials shows that $q_1(z) = P_M(z)$ and $q_2(z) = P_M(1-z)$ are the uniquely determined polynomials $q_1(z)$ and $q_2(z)$ with degrees $\leq M$ for which a Bezout identity

$$(1-z)^{M+1}\, q_1(z) + z^{M+1}\, q_2(z) = 1$$

  holds

- But these are, even without bounding the degrees, this is the only solutions of this equation!
  - For any solution $(q_1(z), q_2(z))$ one must have the relation $q_2(z) = q_1(1-z)$
    (Write down the Bezout identity again, but with z replaced by $1-z$, and then subtract both identities)
  - From
    $$(1-z)^{M+1}\, q(z) + z^{M+1}\, q(1-z) = 1,$$
    one has
    $$q(z) = P_M(z) + a(z)\, z^{M+1}, \quad q(1-z) = P_M(1-z) - a(z)(1-z)^{M+1},$$
    for some polynomial $a(z)$,
  - which only holds for the zero polynomial

- Now write the BEZOUT identity in the following way

$$P_M(z) = (1-z)^{-M-1} - \left(\frac{z}{1-z}\right)^{M+1} \cdot P_M(1-z),$$

and take the series development

$$(1-z)^{-M-1} = \sum_{m \geq 0} \binom{M+m}{m} z^m$$

into account. By developing both sides one gets the explicit form of the Daubechies polynomials because on the left-hand side one has a polynomial of degree $\leq M$, and the second term on the right-had side only contributes to z-powers of degrees $> M$

- Construction of symmetric filters of odd length
  - Let $\boldsymbol{h} = (h_{-L}, \ldots, h_L)$ be a symmetric filter of length $2L+1$, so that its Fourier series $H(\omega) = \sum_{k=-L}^{L} h_k e^{ik\omega}$ is an even function

$$H(\omega) = h_0 + 2\sum_{k=1}^{L} h_k \cos(k\omega)$$

  - For $k \in \mathbb{Z}$ the term $\cos(k\omega)$ can be written as a polynomial of degree k in $\cos(\omega)$, thus $H(\omega)$ is a polynomial of degree L in $\cos(\omega)$
  - From the low-pass condition

$$H(0) = \sqrt{2}, H(\pi) = H'(\pi) = \ldots = H^{(\ell)}(\pi) = 0, H^{(\ell+1)} \neq 0$$

one gets

$$H(\omega) = \sqrt{2}\,(1 + \cos(\omega))^{\ell}\, q(\cos(\omega)),$$

where $q(z)$ is a polynomial of degree $L - \ell$ which satisfies $q(\cos(\pi)) = q(-1) \neq 0$

- Construction of symmetric filters of odd length (seq.)
  - From $H(0) = \sqrt{2}$ one gets $q(1) = 2^{-\ell}$
  - Replacing now $1 + \cos(\omega)$ by $2\cos^2(\omega/2)$, one obtains

$$H(\omega) = \sqrt{2}\,\cos^{2\ell}(\omega/2)\,p(\cos(\omega)),$$

where $p(z)$ is a polynomial of degree $L - \ell$ with $p(1) = 1$ and $p(-1) \neq 0$

Proposition

If $\boldsymbol{h}$ and $\widetilde{\boldsymbol{h}}$ are symmetric filters of odd length with Fourier series

$$H(\omega) = \sqrt{2}\,\cos^{2\ell}(\omega/2)\,p(\cos(\omega)),$$
$$\widetilde{H}(\omega) = \sqrt{2}\,\cos^{2\widetilde{\ell}}(\omega/2)\,\widetilde{p}(\cos(\omega)),$$

satisfying the orthogonality condition

$$H(\omega)\,\widetilde{H}(\omega) + H(\omega+\pi)\,\widetilde{H}(\omega+\pi) = 2,$$

then (with $K = \ell + \widetilde{\ell}$) one has

$$p(\cos(\omega)) \cdot \widetilde{p}(\cos(\omega)) = P_{K-1}(\sin^2(\omega/2))$$

- About the proof:
  - Substituting into the orthogonality condition gives

$$\cos^{2K}(\omega/2)\, p(\cos(\omega))\, \widetilde{p}(\cos(\omega)) + \sin^{2K}(\omega/2)\, p(-\cos(\omega))\, \widetilde{p}(-\cos(\omega)) = 2$$

  - Set $P(z) = p(z)\, \widetilde{p}(z)$, then $P(\cos(\omega))$ is a polynomial in $y = \sin^2(\omega/2)$, so that writing $\widehat{P}(y)$ for $P(\cos(\omega))$ the orthogonality relation turns into

$$(1-y)^K \widehat{P}(y) + y^K \widehat{P}(1-y) = 1,$$

    which identifies $\widehat{P}(y)$ as a Daubechies polynomial

Constructing the Cohen-Daubechies-Feauveau-7/9 filter pair

- Start with the Daubechies polynomial

$$P_3(z) = \binom{3}{0} + \binom{4}{1} z + \binom{5}{2} z^2 + \binom{6}{3} z^3 = 1 + 4z + 10z^2 + 20z^3$$

- The 3 complex roots of this polynomial can be determined exactly

$$z_1 = \frac{1}{6}\left(-1 - \frac{7^{2/3}}{\sqrt[3]{5\left(3\sqrt{15}-10\right)}} + \frac{\sqrt[3]{7\left(3\sqrt{15}-10\right)}}{5^{2/3}}\right)$$

$$z_2 = -\frac{1}{6} + \frac{7^{2/3}\left(1+i\sqrt{3}\right)}{12\sqrt[3]{5\left(3\sqrt{15}-10\right)}} - \frac{\left(1-i\sqrt{3}\right)\sqrt[3]{7\left(3\sqrt{15}-10\right)}}{12\ 5^{2/3}}$$

$$z_3 = -\frac{1}{6} + \frac{7^{2/3}\left(1-i\sqrt{3}\right)}{12\sqrt[3]{5\left(3\sqrt{15}-10\right)}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{7\left(3\sqrt{15}-10\right)}}{12\ 5^{2/3}}$$

- It suffices to take approximate values

$$z_1 \approx -0.342384$$
$$z_2 \approx -0.078808 + 0.373931i$$
$$z_3 \approx -0.078808 - 0.373931i$$

- The polynomial $P_3(z)$ factors into two polynomials

$$p(z) = a \cdot (z - z_1)$$
$$\widetilde{p}(z) = \frac{1}{a} \cdot (z - z_2)(z - z_3)$$

where the constant a has to be determined

- In terms of approximate values

$$p(z) \approx a \cdot (z + 0.342384)$$
$$\widetilde{p}(z) \approx \frac{1}{a}(z + 0.078808 - 0.373931i)(z + 0.078808 + 0.373931i)$$
$$\approx \frac{1}{a}(2.9207 + 3.15232z + 20z^2)$$

- The two filters $\boldsymbol{h} = (h_j)_{j=-3..3}$ and $\widetilde{\boldsymbol{h}} = (\widetilde{h}_j)_{j=-4..4}$ are defined through their frequency representations (note that $K = 4, \ell = \widetilde{\ell} = 2$)

$$\begin{aligned} H(\omega) &= \sqrt{2}\cos(\omega/2)^4 p(\sin(\omega/2)^2) \\ &= a \cdot \sqrt{2}\cos(\omega/2)^4 \left(0.342384 + \sin(\omega/2)^2\right) \\ \widetilde{H}(\omega) &= \sqrt{2}\cos(\omega/2)^4 \widetilde{p}(\sin(\omega/2)^2) \\ &= \frac{1}{a}\cos(\omega/2)^4 \left(4.13049 + 4.45805\sin(\omega/2)^2 + 20\sqrt{2}\sin(\omega/2)^4\right) \end{aligned}$$

- Now the value of a can be fixed by requiring $H(0) = \sqrt{2}$ (and also $\widetilde{H}(0) = \sqrt{2}$), which gives

$$a = 2.9207$$

- so that

$$\begin{aligned} H(\omega) &= 4.13049\cos(\omega/2)^4 \left(0.342384 + \sin(\omega/2)^2\right) \\ \widetilde{H}(\omega) &= \cos(\omega/2)^4 \left(1.41421 + 1.52637\sin(\omega/2)^2 + 9.68408\sin(\omega/2)^4\right) \end{aligned}$$

- Converting the sin- and cos-expressions into exponentials then gives the filter coefficients

$$(h_j)_{j=-3..3} = \begin{bmatrix} -0.0645388826 \\ -0.0406894175 \\ 0.4180922731 \\ 0.7884856164 \\ 0.4180922731 \\ -0.0406894175 \\ -0.0645388826 \end{bmatrix} \quad (\widetilde{h}_j)_{j=-4..4} = \begin{bmatrix} 0.0378284555 \\ -0.0238494650 \\ -0.1106244044 \\ 0.3774028555 \\ 0.8526986788 \\ 0.3774028555 \\ -0.1106244044 \\ -0.0238494650 \\ 0.0378284555 \end{bmatrix}$$

- Low-pass properties: from the definition it is clear that both filters $\boldsymbol{h} = (h_j)_{j=-3..3}$ and $\widetilde{\boldsymbol{h}} = (\widetilde{h}_j)_{j=-4..4}$ have 4 vanishing moments, i.e., they have very good smoothness properties for reconstruction

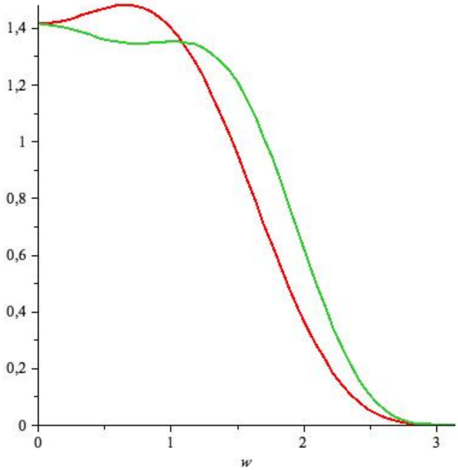


Figure: Frequency picture of the Cohen-Daubechies-Feauveau-(7,9) filter pair

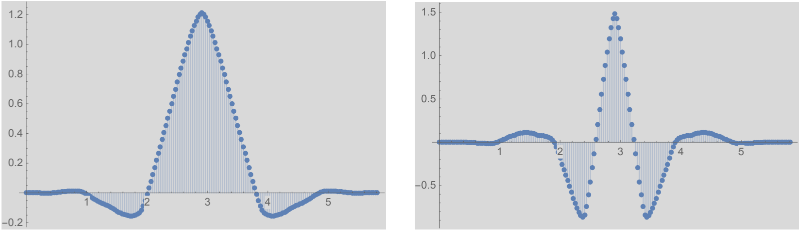


Figure: Scaling and wavelet functions for the CDF-7 filter

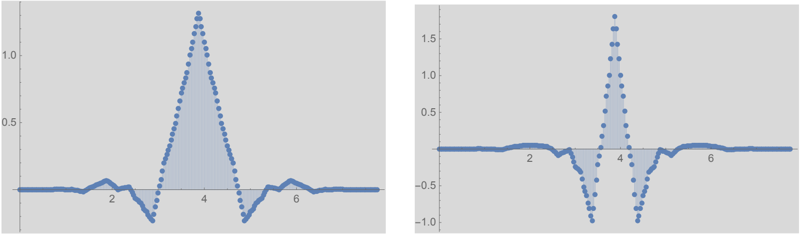


Figure: Scaling and wavelet functions for the CDF-9 filter