Pre-processing

Non-Linear Filtering

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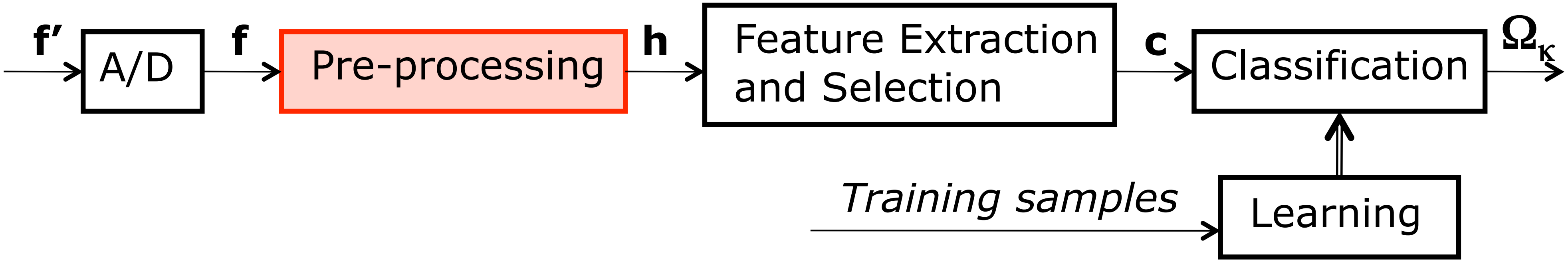
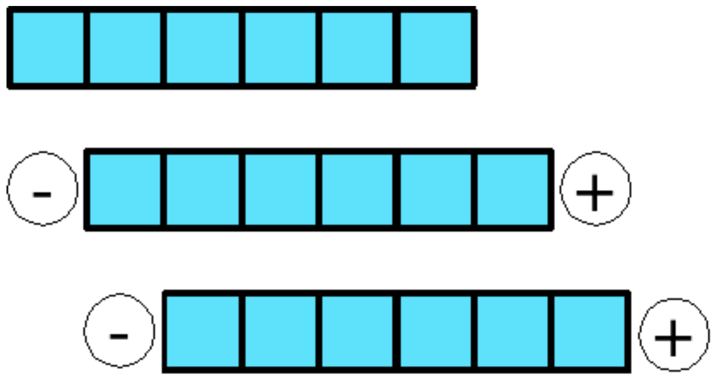
Pattern Recognition Pipeline



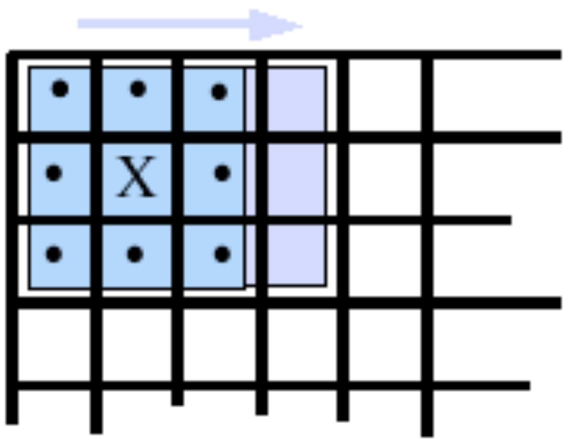
- The goal of pre-processing is to transform a signal f to another signal h so that the resulting signal h
  - makes subsequent processing easier
  - makes subsequent processing better (more accurate)
  - makes subsequent processing faster
- Already studied *histogram equalization, thresholding*, *smoothing* and *edge detection*.

Recursive Filter

- A recursive filter is a type of filter which re-uses one or more of its outputs in one step as an input in the subsequent computation step.
- Example: Moving Average Filter



1D moving average



2D moving average

Move one step (pixel) to the right, cancel the leftmost value (or column of values in the case of 2D window) and add the values in rightmost position (or column).

Formulation of a Recursive Filter

- Recall that convolution in 1D is defined as:

$$R(j) = \sum_{\mu} h(\mu) f(j-\mu) \Leftrightarrow R_j = \sum_{\mu} h_\mu f_{j-\mu}$$

- The idea is then to evaluate R_j using R_{j-1} or even older evaluations in computing the current position.

$$R_0 = f_0$$

$$R_j = (1-\alpha) R_{j-1} + \alpha f_j$$

- But (j-1) was itself built using (j-1)-1. So another more general formulation of the recursive filter is:

$$R_j = \sum_{\mu=0}^{n-1} \alpha_\mu f_{j-\mu} - \sum_{\mu=1}^{m} \beta R_{j-\mu}$$

Comments on Recursive Filtering

- Deriving a recursive definition of a filter is usually non-trivial.
- However, the speedup can often be significant and is well-worth the effort in real life applications.
- The derivation is performed once. The benefit of the speedup is achieved every time the recursive filter is used.
- Prof. Neimann’s textbook also has the derivation of the recursive definition of the Gaussian filter.
- Other examples of recursive filtering: Kalman filtering and Particle Filtering (though they perform a different type of “filtering”).

Homomorphic Filtering

- So far we have only looked at Linear Shift Invariant systems. But there are a number of filters that do not fall under this category.
- We need a more general system than a linear system.
- **Homomorphic systems** are a generalization of linear systems.
- Filters that perform homomorphic transformations are called homomorphic filters.

Homomorphic Map

- A **homomorphic map**, also known as a homomorphism, is a structure-preserving map between the members of two algebraic systems (e.g. two vector spaces, two groups, two rings, two fields, etc.).

- The elements and the operations between the two systems **may appear entirely different,** but results from one system, apply on the other.

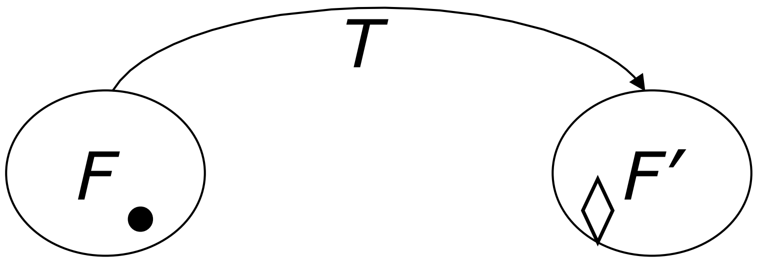
Definition of Homomorphic Map

- Given two spaces *F* and *F′*, where $(f_1, f_2, \ldots) \in F$ and $(f_1', f_2', \ldots) \in F'$, and a binary operation defined on each of them, then a map *T*, between the two spaces, $T : F \to F'$ is *homomorphic* if:

$$T(f_1 \bullet f_2) = T(f_1) \lozenge T(f_2) \qquad \forall f_1, f_2 \in F$$

$\bullet$ is a binary operation defined in F

$\lozenge$ is a binary operation defined in F'



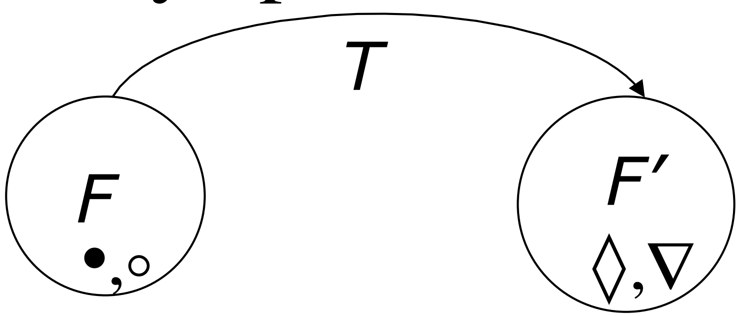
Definition of Homomorphic Map - continued

- Linear systems is a *special case* of homomorphic systems where the operations on each space are the same, i.e. $\bullet = \lozenge = +$
- Prof. Neimann in his book specifies a 2nd property of homomorphic maps:

$$T(\alpha \circ f_1) = \alpha \nabla T(f_1) \qquad \forall f_1 \in F \quad \text{and} \quad \forall \alpha \in \Re$$

$\circ$ is a unary operation defined in F

∇ is a unary operation defined in F'



Examples of Homomorphic Maps

- The logarithm is an example of a homomorphic map:

$$T(f_1 \bullet f_2) = T(f_1) \lozenge T(f_2) \qquad \forall f_1, f_2 \in F$$

$$\log(f_1 f_2) = \log(f_1) + \log(f_2) \qquad \forall f_1, f_2 \in F$$

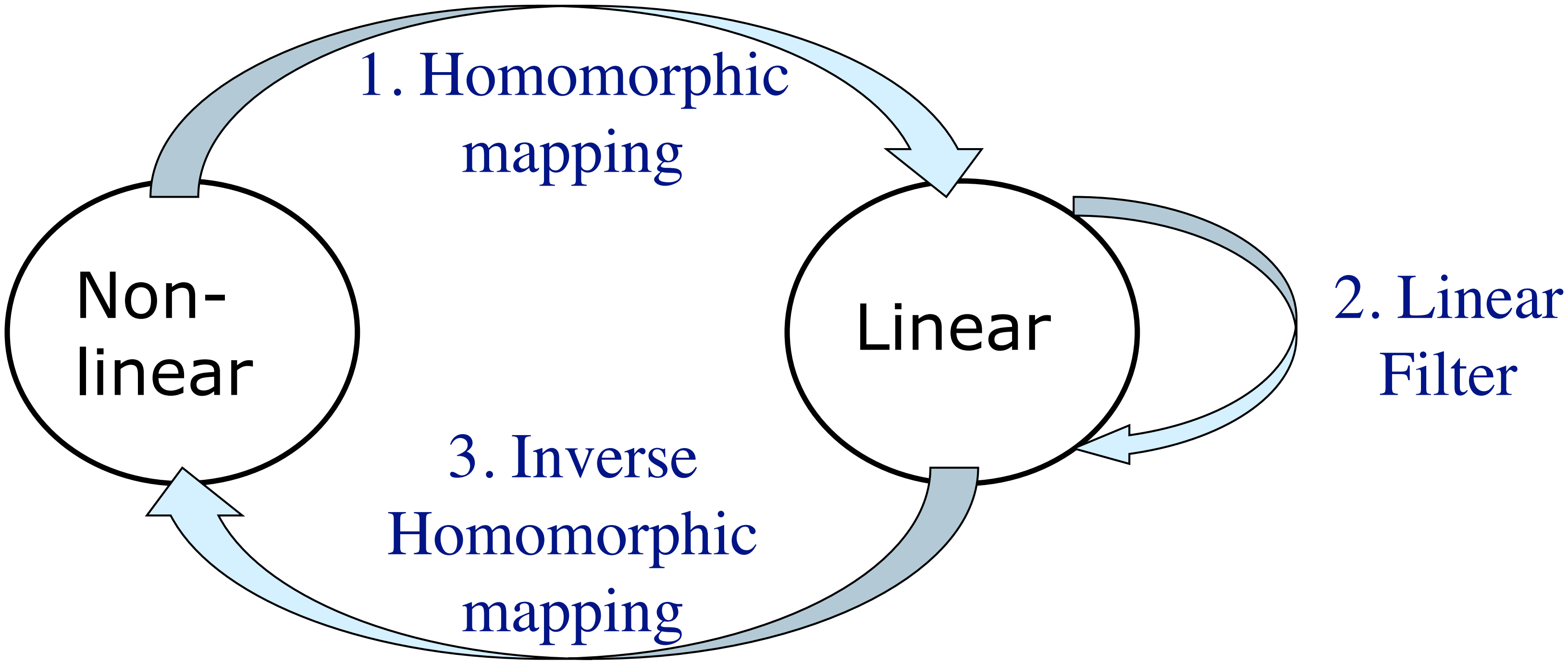
- Note that Prof. Neimann's addendum is also satisfied:

$$T(\alpha \circ f_1) = \alpha \nabla T(f_1) \qquad \forall f_1 \in F \ \text{ and } \ \forall \alpha \in \Re^+$$

$$\log(f_1^{\alpha}) = \alpha \log(f_1) \qquad \forall f_1 \in F \ \text{ and } \ \forall \alpha \in \Re^+$$

Homomorphic Filtering Revisited

- The idea behind homomorphic filtering is to map a nonlinear operation to a different domain in which linear filter techniques are applied, followed by mapping back to the original domain.



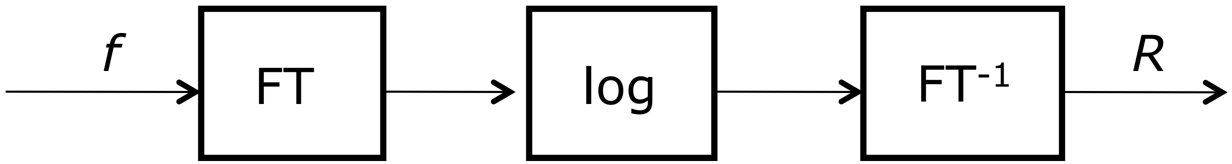
Example: Multiplicative Noise

- The LSI filters that we have seen so far for reducing noise assume additive noise.
- In MRI we can get inhomogeneities that are modelled with multiplicative noise.
- Idea: Use the Logarithm homomorphic map.
- If you have an algorithm for eliminating additive noise (e.g. mean filtering), then you can apply this method to filter a signal with multiplicative noise as follows:

$$f_{noisy} = f_{clean} n \Rightarrow \log(f_{noisy}) = \log(f_{clean}) + \log(n)$$

Cepstrum

- Another homomorphic filter that is based on the Fourier Transform (FT) is the so called **cepstrum**.
- It is used in speech recognition.
- Given an image *f*, apply the FT, then apply the log, and then the inv. FT.



- This series of operations is called the cepstrum.

$$T_{cepstrum}(f) = \mathrm{FT}^{-1}(\log(\mathrm{FT}(f)))$$

- For speech signals we used the complex logarithm and thus have the complex cepstrum.

Cepstrum - continued

- The cepstrum holds information about the rate of change in the different spectrum bands.
- It was originally used for characterizing the seismic echoes resulting from earthquakes or explosions.
- The power cepstrum is defined as

$$T_{Power\_cepstrum}(f) = \left|\mathrm{FT}(\log(\left|\mathrm{FT}(f)\right|^2))\right|^2$$

- It is often used as a feature vector in sound/speech analysis, especially in voice identification and pitch analysis. It is very good at separating the effects of vocal cord vibration from formant filtering in the vocal tract.

Mathematical Morphology

- There is a family of non-linear operations that one can perform on images, which is based on a class of mathematics called mathematical morphology.
- The field of mathematical morphology provides a collection of tools/operations for the systematic analysis of shape.
- Mathematical morphology is based on set theory, topology and lattice algebra.
- Thus, in order to understand how morphological operations work, we must first express images as sets.

Images as Sets

- An image can be treated as a set of tuples {(Coordinate, Value)}. For example, the function f(x) can be represented by the set:

$$A = \{(x_1, f_1), (x_2, f_2), \ldots (x_n, f_n)\}$$

- For 2D data we have:

$$A = \{(x_1, y_1, f_1), (x_2, y_2, f_2), \ldots (x_n, y_n, f_n)\}$$

- For a binary image, white pixels are normally taken to represent foreground regions (value 1), while black pixels, (value 0), denote background.
- Then the set representation of the binary image is simply the set of the 2D coordinates of all the foreground pixels in the image, i.e. of all the pixels with value 1.

$$A = \{(x, y) \mid f(x, y) = 1\}$$

Morphological Operations

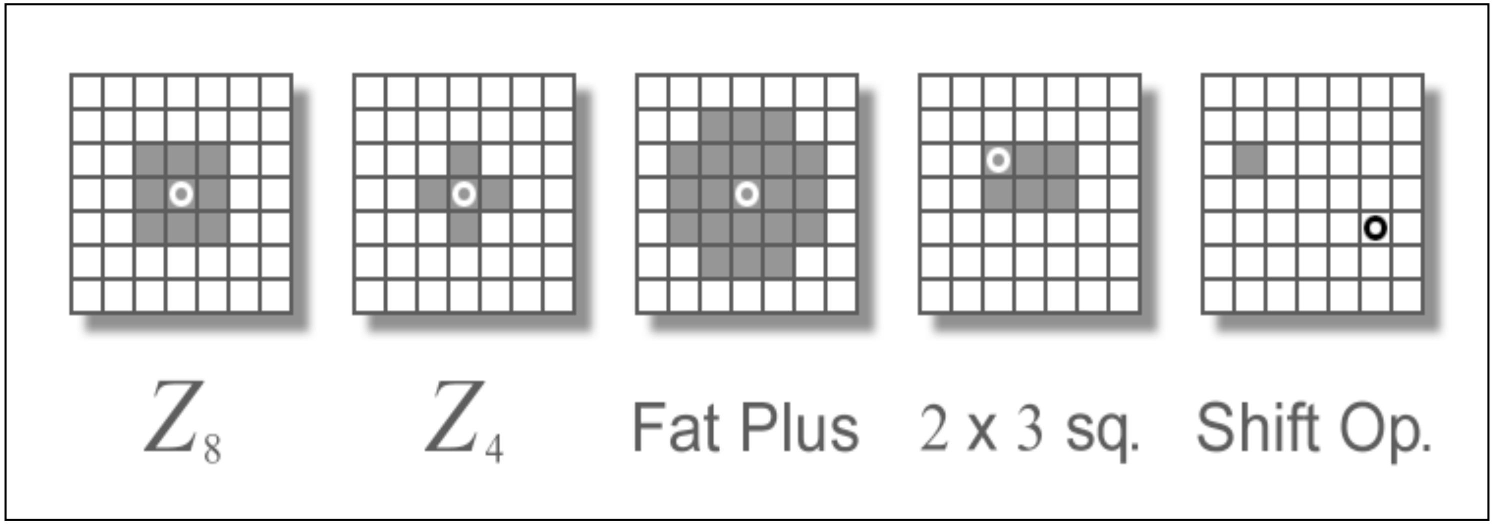
- When defining a morphological operation one needs to specify:

  1. The input image - set A
  2. The structuring element - set S
  3. The set operation(s) performed on sets A and S.

Structuring Element

- The structuring element, S, is a small image.

Example SEs



- S can be of any shape, size or connectivity. In this figure, gray pixels are foreground and white are background. The circles indicate the position of the origin.

Figure courtesy of R. A. Peters II

Binary Erosion

- Idea: If part of the structuring element lies on the background remove the pixel at the origin of the structuring element from the foreground.

$$R = A \ominus S = \{ p \mid (S)_p \subset A \}$$

- R is a new image where a pixel p is set to 1 iff: when the origin of S is positioned at p, i.e. $(S)_p$, S is a proper subset of A.

Simple Erosion Example

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0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

$\ominus$

1
1
1

Simple Erosion Example

0 (1)	0	0	0	0	0	0	0	0	0	0	0	0
0 (1)	0	1	0	0	0	0	0	0	0	0	0	0
0 (1)	0	0	0	0	0	0	0	0	1	0	0	0
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0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
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0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
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0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

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0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
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Simple Erosion Example

0	0	0	0 (1)	0	0	0	0	0	0	0	0	0
0	0	1	0 (1)	0	0	0	0	0	0	0	0	0
0	0	0	0 (1)	0	0	0	0	0	1	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0

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0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
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Simple Erosion Example

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0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0 (1)	0	0	0	0	0	0	0
0	0	1	0	0	0 (1)	0	0	0	0	0	0	0
0	0	0	0	0	0 (1)	0	0	0	1	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0

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0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0
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Simple Erosion Example

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0	0	0	0	0	0	0	0	0	0	0	0	0

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0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
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Simple Erosion Example

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0	0	0	0	0	0	0	0	0	0	0	0	0

→ …

0	0	0	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0 (1)	0	0	0	0
0	0	1	0	0	0	0	0	0 (1)	0	0	0	0
0	0	0	0	0	0	0	0	0 (1)	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ ...

0	0	0	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0 (1)	0	0	0
0	0	1	0	0	0	0	0	0	0 (1)	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ ...

0	0	0	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0 1	0	0
0	0	1	0	0	0	0	0	0	0	0 1	0	0
0	0	0	0	0	0	0	0	0	1	0 1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ ...

0	0	0	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0 (1)	0
0	0	1	0	0	0	0	0	0	0	0	0 (1)	0
0	0	0	0	0	0	0	0	0	1	0	0 (1)	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0 1
0	0	1	0	0	0	0	0	0	0	0	0	0 1
0	0	0	0	0	0	0	0	0	1	0	0	0 1
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0 1	0	1	0	0	0	0	0	0	0	0	0	0
0 1	0	0	0	0	0	0	0	0	1	0	0	0
0 1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0 1	1	0	0	0	0	0	0	0	0	0	0
0	0 1	0	0	0	0	0	0	0	1	0	0	0
0	0 1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0 1	0	0	0	0	0	0	0	0	0
0	0	0	0 1	0	0	0	0	0	1	0	0	0
0	0	0	0 1	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0 1	0	0	0	0	0	0	0	0
0	0	0	0	0 1	0	0	0	0	1	0	0	0
0	0	0	0	0 1	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0 (1)	0	0	0	0	0	0	0
0	0	0	0	0	0 (1)	0	0	0	1	0	0	0
0	0	0	0	0	0 (1)	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0 / 1	0	0	0	0	0	0
0	0	0	0	0	0	0 / 1	0	0	1	0	0	0
0	0	0	0	0	0	0 / 1	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0 1	0	0	0	0
0	0	0	0	0	0	0	0	0 1	1	0	0	0
0	0	0	0	0	0	0	0	0 1	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0 / 1	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0 / 1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ ...

0	0	0	0	0	0
0	0	0	0	0	0
0	0	**0**	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

…

0	0	0	0	0	0
0	0	0	0	0	0
0	0	**0**	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0 1	0
0	0	0	0	0	0	0	0	0	1	0	0 1	0
0	0	0	0	0	0	0	0	0	0	0	0 1	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	**0**	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0/1
0	0	0	0	0	0	0	0	0	1	0	0	0/1
0	0	0	0	0	0	0	0	0	0	0	0	0/1
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ ...

0	0	0	0	0	0
0	0	0	0	0	0
0	0	**0**	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0 (1)	0	0	0	0	0	0	1	0	0	0
0	0	0 (1)	0	0	0	0	0	0	0	0	0	0
0	0	0 (1)	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0 1	0	0	0	0	1	0	0	0
0	0	0	0	0 1	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	**0**	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0 1	0	0	0	0	0	0	0	0	0	0
0	0	0 1	1	1	1	1	1	1	1	0	0	0
0	0	0 1	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	**0**	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	**0**	**0**	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	**0**	**0**	**0**	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0 1	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	**0**	**0**	**0**	**0**	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	**0**	**0**	**0**	**0**	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	**0**	**0**	**0**	**0**	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	**0**	**0**	**0**	**0**	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	**0**	**0**	**0**	**0**	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	**0**	0	0	0
0	0	0	0	0	0
0	**0**	**0**	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0/1	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0/1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	**0**	0	0	0
0	0	0	0	0	0
0	**0**	**0**	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	**0**	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	**0**	0	0	0
0	0	0	0	0	0
0	**0**	**0**	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	**0**	0	0
0	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0 1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	**0**	0	0	0
0	0	0	0	0	0
0	**0**	**0**	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	**0**	0	0
0	**0**	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	0	0	0	0
0	0	**0**	0	0	0
0	0	0	0	0	0
0	**0**	**0**	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	**0**	0	0
0	**0**	**0**	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

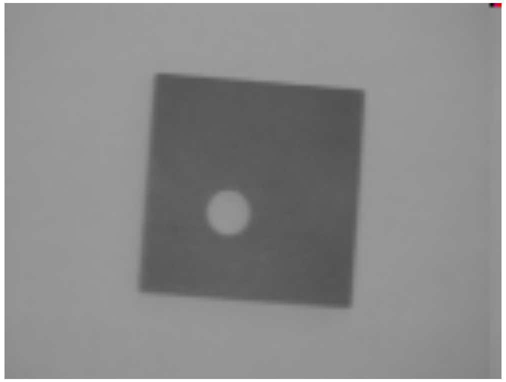
End Result of Simple Erosion Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	**0**	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	**0**	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	**0**	0	0
0	0	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	**0**	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

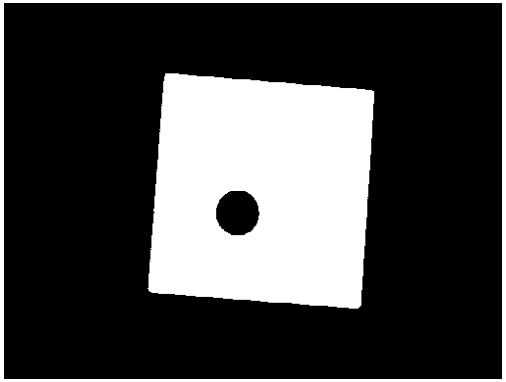
Real Image Erosion Example

Structuring Element, S =

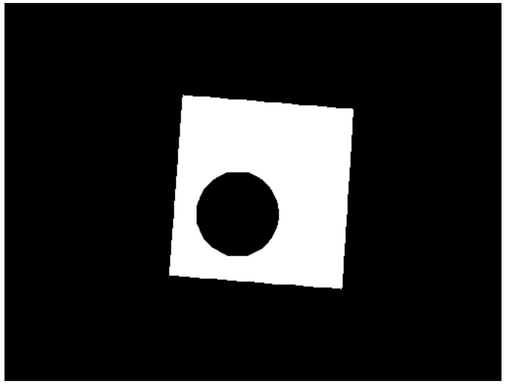
1	1	1
1	1	1
1	1	1



Original image



Binary image



Eroded image

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Binary Dilation

- Idea: If part of the structuring element lies on the foreground add the pixel at the origin of the structuring element to the foreground.

$$R = A \oplus S = \{ p \,|\, ((S)_p \cap A\}) \neq \varnothing \}$$

- R is a new image where a pixel p is set to 1 iff: when the origin of S is positioned at p, i.e. $(S)_p$, the intersection of A with S is not empty.

Simple Dilation Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

$\oplus$

1
1
1

Simple Dilation Example

0/1	0	0	0	0	0	0	0	0	0	0	0	0
0/1	0	1	0	0	0	0	0	0	0	0	0	0
0/1	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Dilation Example

0	0 (1)	0	0	0	0	0	0	0	0	0	0	0
0	0 (1)	1	0	0	0	0	0	0	0	0	0	0
0	0 (1)	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Dilation Example

0	0	0 1	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0 1	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Dilation Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	0	**1**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Dilation Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0 1	0	0	0	0	0	0	1	0	0	0
0	0	0 1	0	0	0	0	0	0	0	0	0	0
0	0	0 1	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	0	**1**	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Dilation Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0/1	0	0	0	0	0	1	0	0	0
0	0	0	0/1	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	0	**1**	0	0	0	0	0	0
0	0	0	**1**	0	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Dilation Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0 1	0	0	0	0	1	0	0	0
0	0	0	0	0 1	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	0	**1**	0	0	0	0	0	0
0	0	0	**1**	**1**	0	0	0	0
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Simple Dilation Example

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0 1	0	0
0	0	0	1	1	1	1	1	1	1	0 1	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

→ . . .

0	0	0	0	0	0
0	0	**1**	0	0	0
0	0	1	0	0	0
1	**1**	**1**	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	**1**	0	0
1	1	1	1	0	0
1	1	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

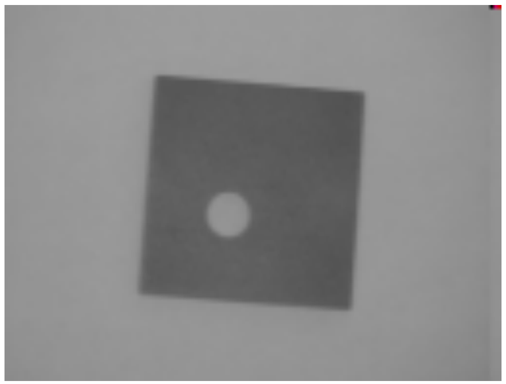
End Result of Simple Dilation Example

0	0	**1**	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	**1**	0	0	0
0	0	**1**	0	0	0	0	0	0	1	0	0	0
0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	1	1	**1**	0	0
0	0	0	1	1	1	1	1	1	1	1	0	0
0	0	0	1	1	1	1	1	1	1	**1**	0	0
0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0
0	0	0	**1**	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	**1**	0	0	0	0	0	0	0	0	0

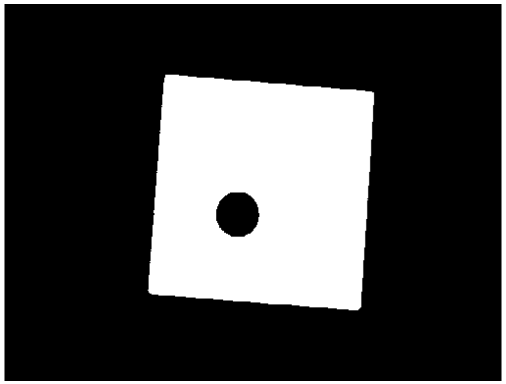
Binary Image Dilation

Structuring Element, S =

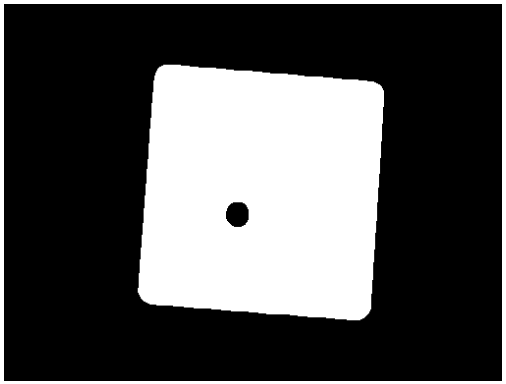
1	1	1
1	1	1
1	1	1



Original image



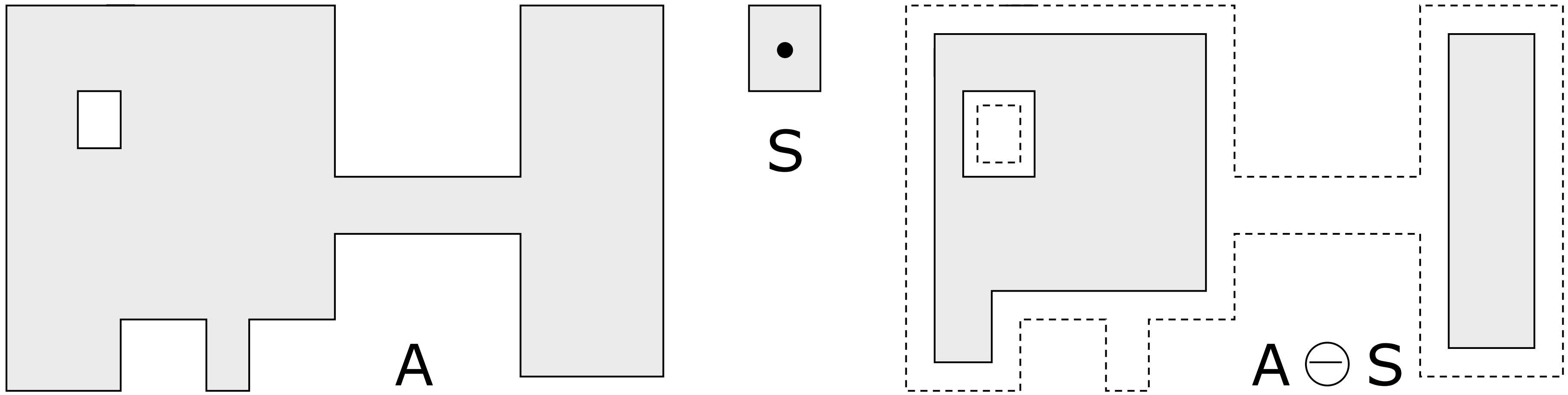
Binary image

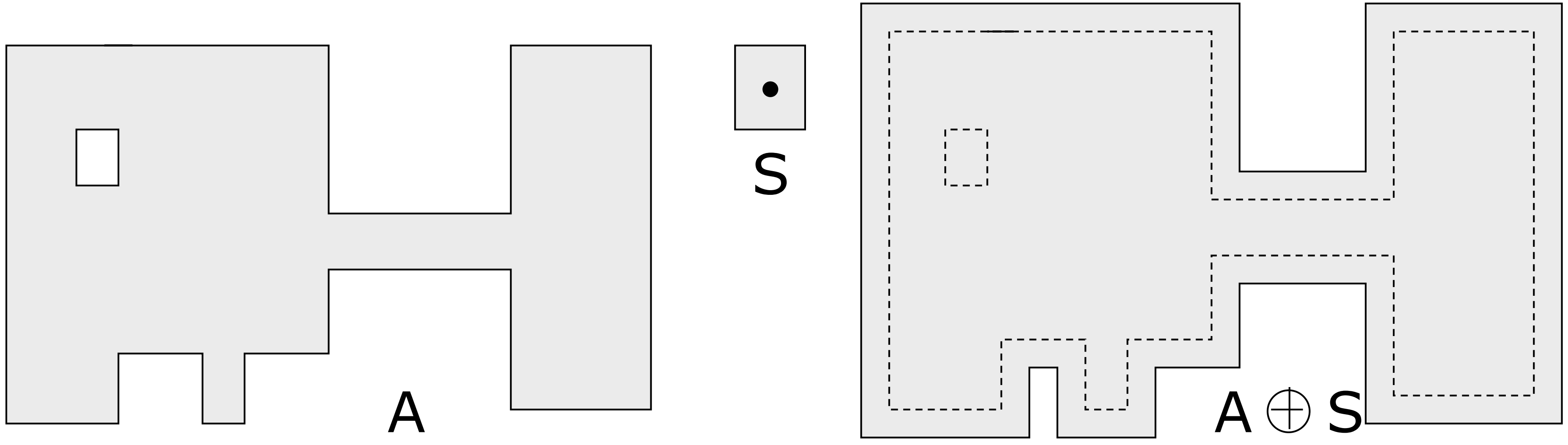


Dilated image

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Schematic Representation of Erosion and Dilation





Binary Opening and Closing

- Opening: An Erosion followed by a Dilation.

$$R = A \circ S = (A \ominus S) \oplus S$$

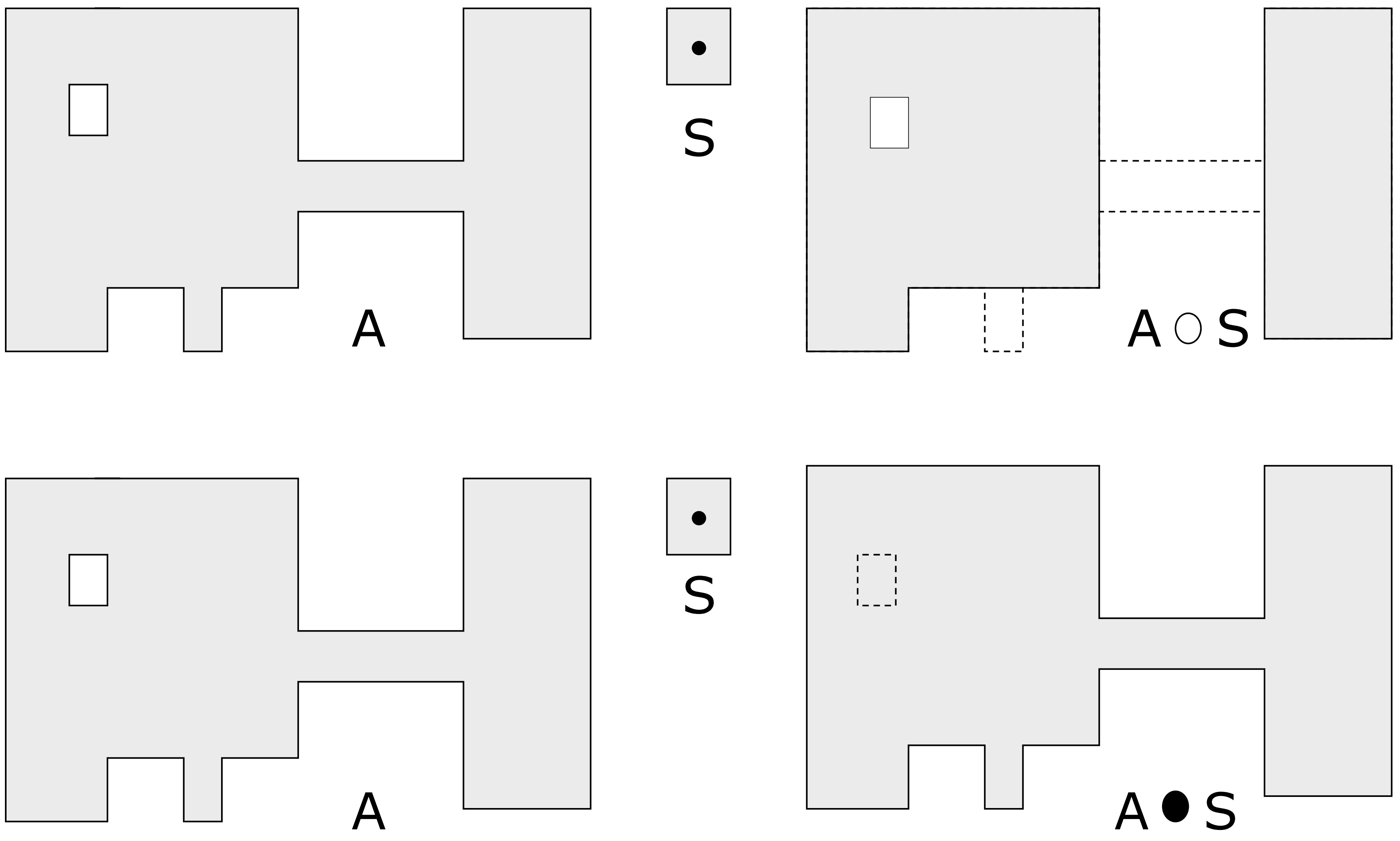
It eliminates spikes and cuts off “bridges”.

- Closing: A Dilation followed by an Erosion.

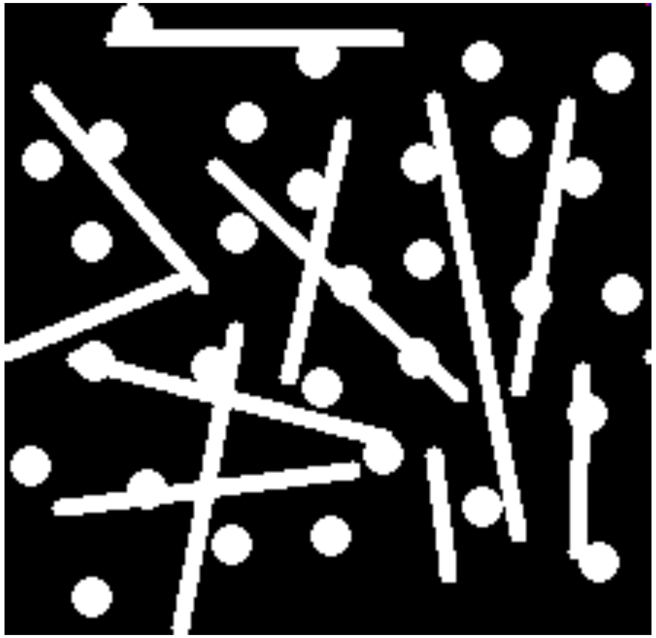
$$R = A \bullet S = (A \oplus S) \ominus S$$

It eliminates bays along the boundaries and fills up holes.

Schematic Representation of Opening and Closing



Binary Image Opening



Original image

Structuring element: circle with diameter, d = 11pixels

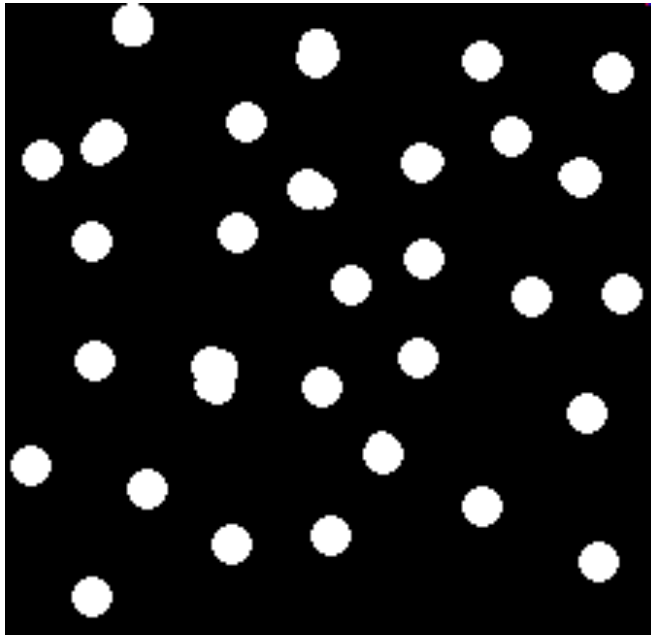
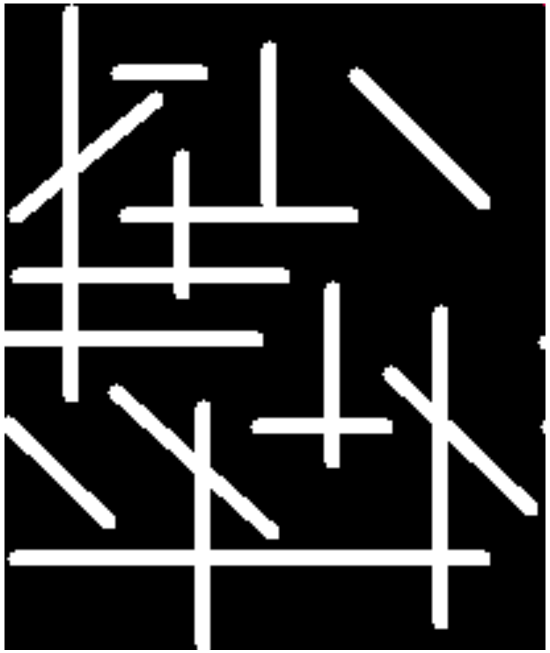


Image after opening

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Binary Image Opening



Original image

Structuring element:
3x9 vertically oriented rectangle

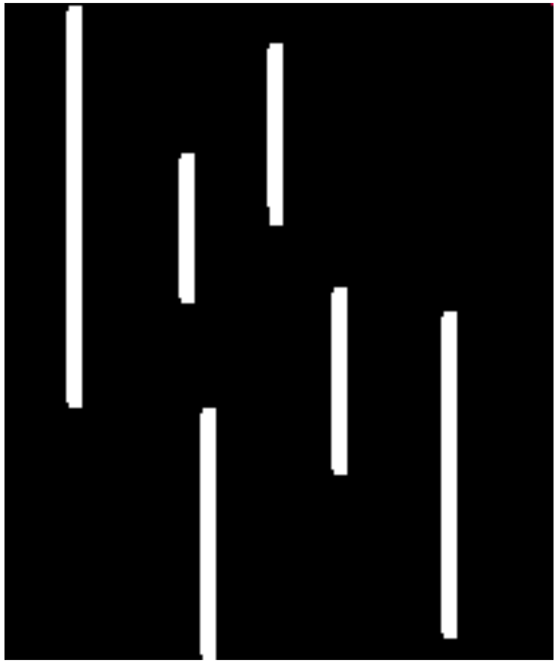
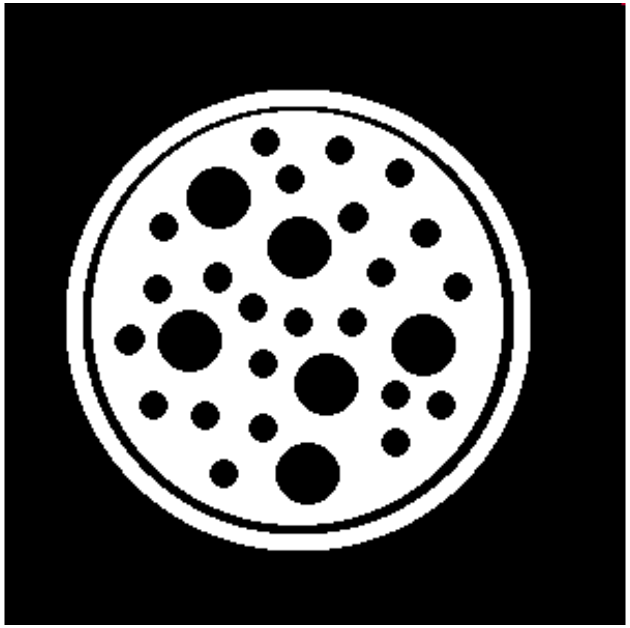


Image after opening

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Binary Image Closing



Original image

Structuring element: circle with diameter, d = 22pixels

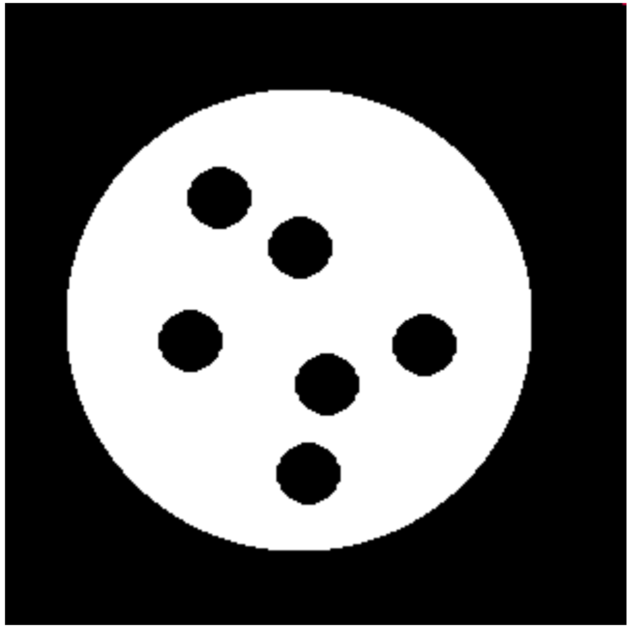


Image after closing

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

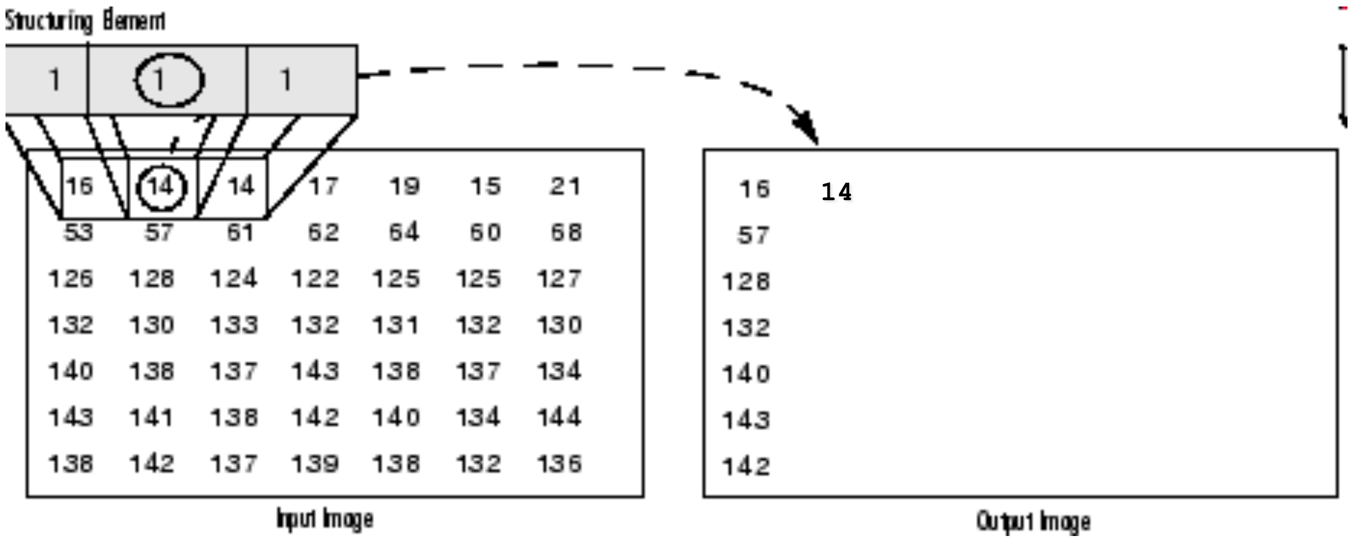
Erosion in Grayscale Images

- In grayscale images erosion is generalized to the minimum operation.
- R is a new image where a pixel p at the origin of the structuring element S is set to the minimum value among all the pixels in the intersection of A and S.

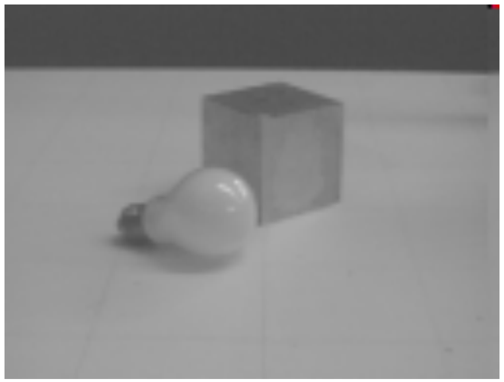
$$A \ominus S: \quad R(p) = \min\left((S)_p \cap A\right)$$

- Erosion decreases the brightness of bright objects against a dark background. Therefore sometimes we refer to erosion as decreasing the size of the object.

Grayscale Image Erosion



Graph adapted from Mathworks



Original image

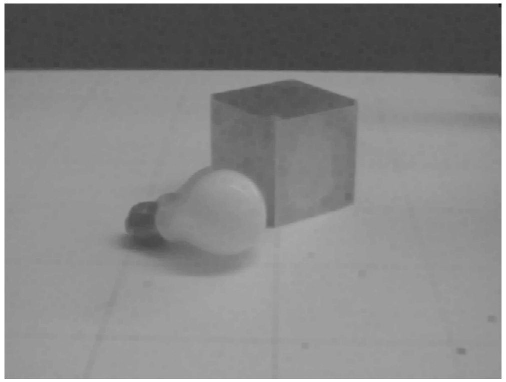


Image eroded once

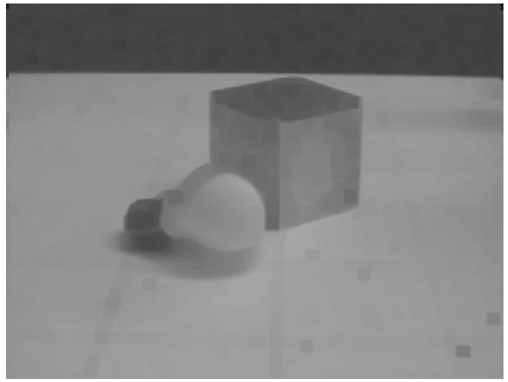


Image eroded twice

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Erosion and “Salt” Noise

Original image with salt noise

Image after erosion

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

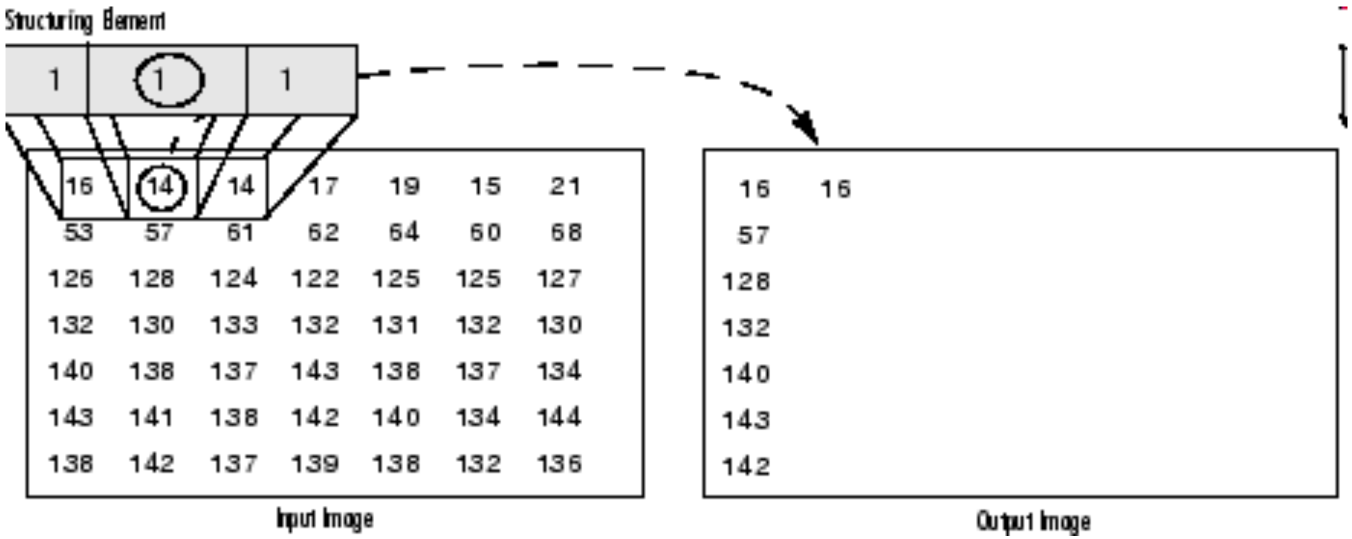
Dilation in Grayscale Images

- In grayscale images dilation is generalized to the maximum operation.
- R is a new image where a pixel p at the origin of the structuring element S is set to the maximum value among all the pixels in the intersection of A and S.

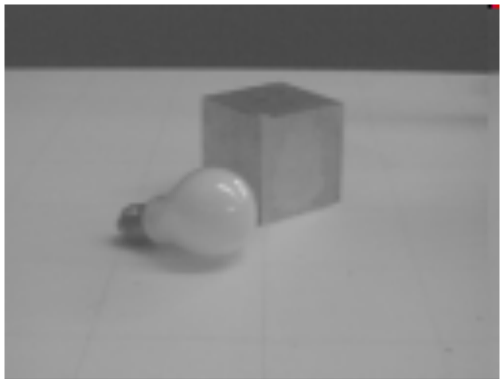
$$A \oplus S: \quad R(p) = \max\left((S)_p \cap A\}\right)$$

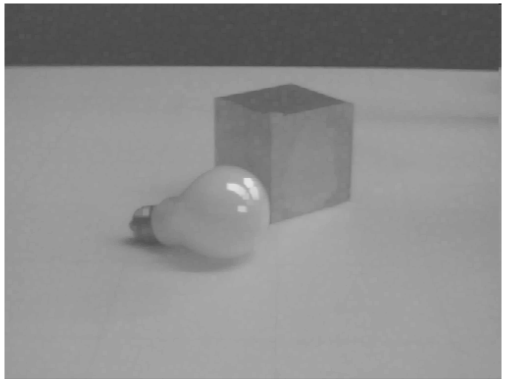
- Dilation increases the brightness of bright objects against a dark background. Therefore sometimes we refer to dilation as increasing the size of the object.

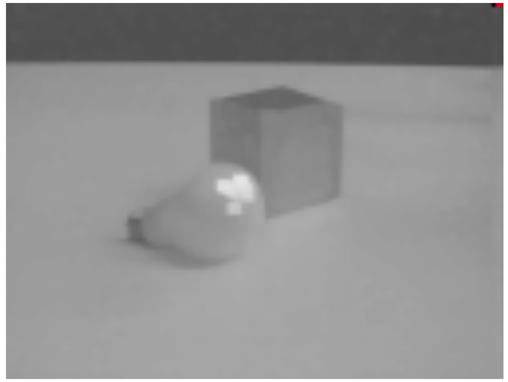
Grayscale Image Dilation



Graph courtesy of Mathworks







Original image   Image dilated once   Image dilated twice

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Dilation and “Pepper” Noise

Original image with pepper noise

Image after dilation

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Opening and "Salt" Noise

Original image with salt noise

Image after opening

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Opening and “Salt” Noise

Original image

After opening

After erosion

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

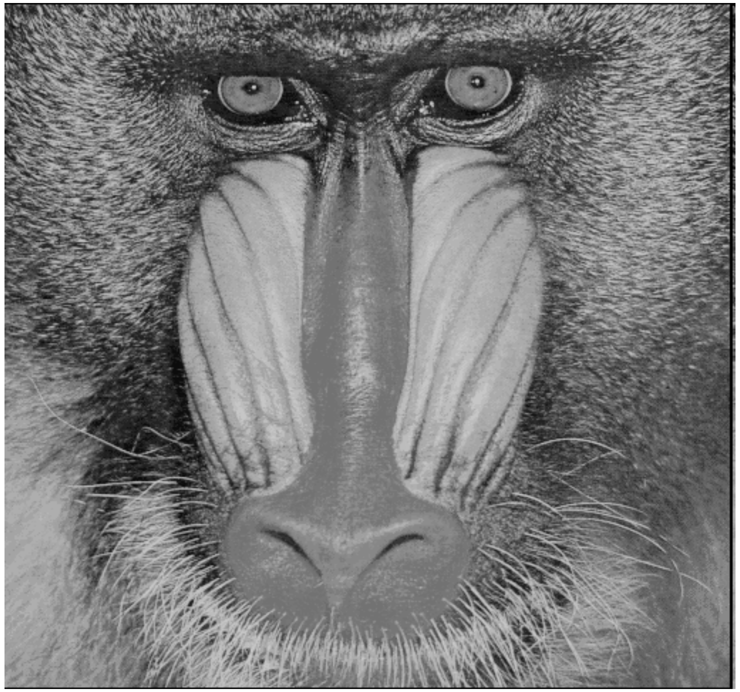
Opening and “Pepper” Noise

Original image with pepper noise

Image after opening

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Grayscale Image Opening



Original image

Structuring element: flat 5x5 square

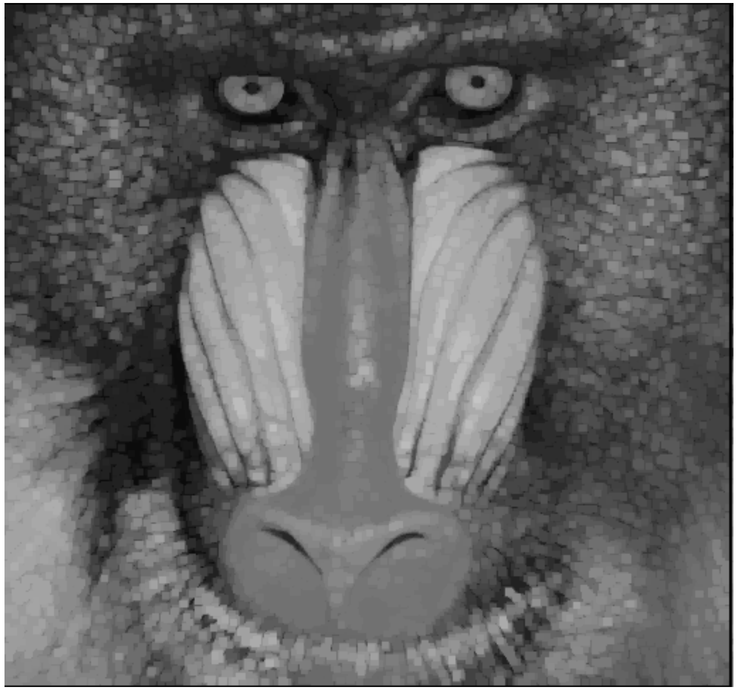


Image after opening

Selective preservation of particular intensity values.

Bright regions which are smaller than the structuring element have been dimmed, while larger ones remain approximately the same.

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Closing and “Pepper” Noise

Original image with pepper noise

Image after closing

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Closing and "Pepper" Noise

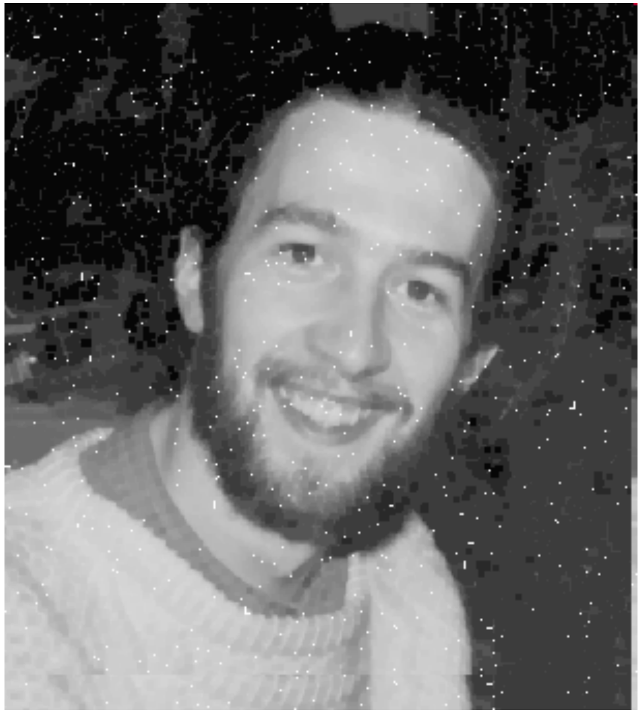
Original image

After closing

After dilation

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Closing and "Salt" Noise



Original image with salt noise

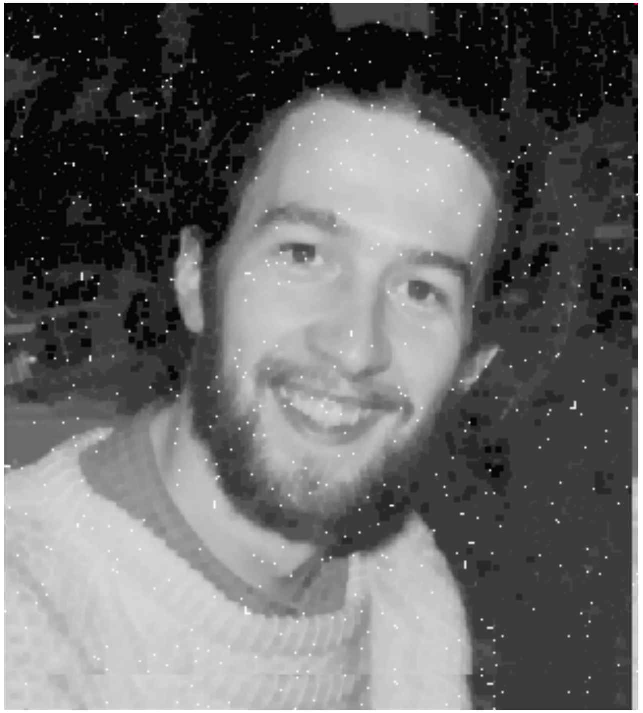
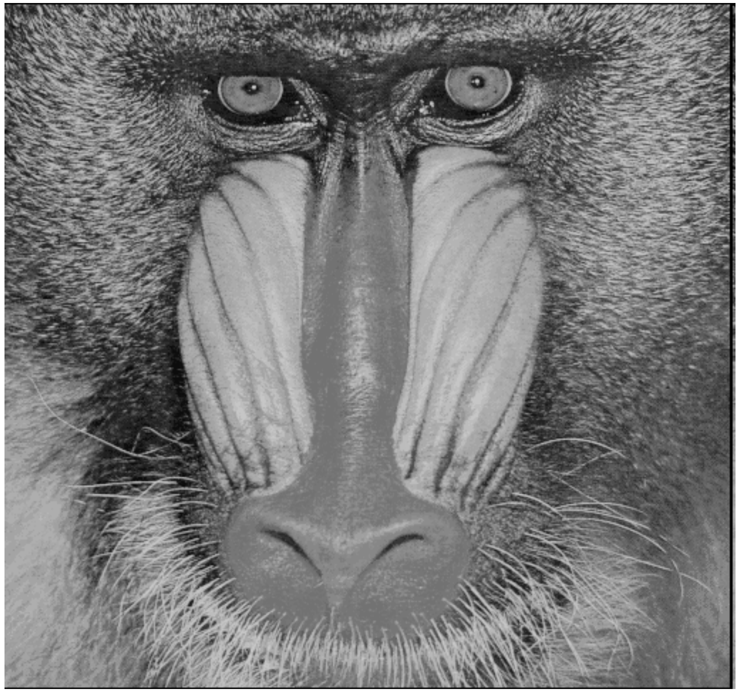


Image after closing

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Grayscale Image Closing



Original image

Structuring element: 5x5 flat square

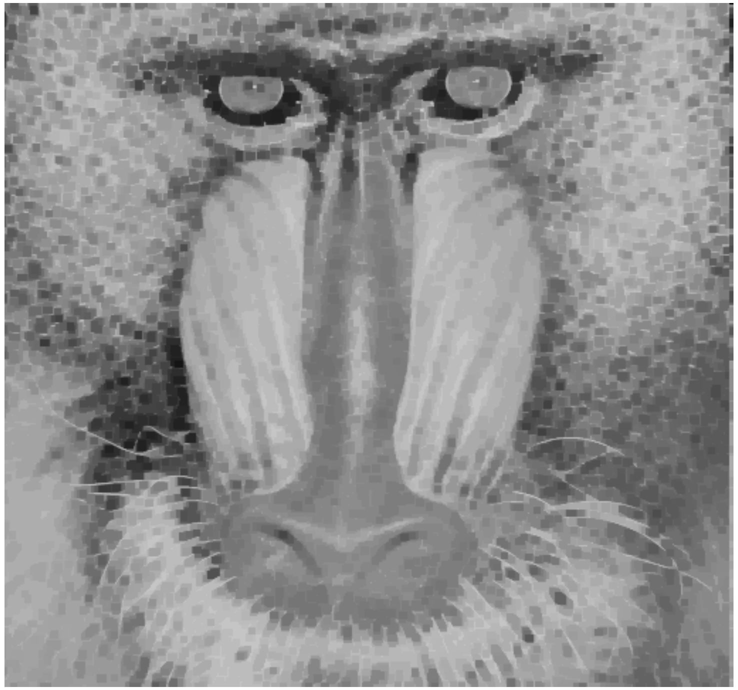


Image after closing

Selective preservation of particular intensity values.

Dark regions which are smaller than the structuring element have been brightened, while larger ones remain approximately the same.

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart

Rank Operations

- Define a local neighborhood $Q = (2N+1) \times (2M+1)$ around a pixel f_{jk}.
- Let all the pixels in that neighborhood Q define a set $W_{f_{jk}}$:

$$W_{f_{jk}} = \{ f_{\mu\nu} \| \mu - j | \le N, |\nu - k| \le M \}$$

- Define an ordering R_{jk} of all the pixels in the local neighborhood, such that:

$$R_{jk} = \{ r_1 \le r_2 \le r_3 \le \ldots \le r_Q \} \text{ where } r_i \in W_{f_{jk}}$$

- A **rank order operation** is a function on a specific order value.

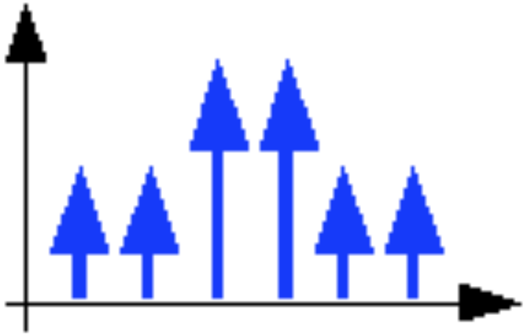
$$h_{jk} = \phi(R_{jk})$$

Rank Operations via Morphology

- Special rank operations $h_{jk} = \phi(R_{jk})$:

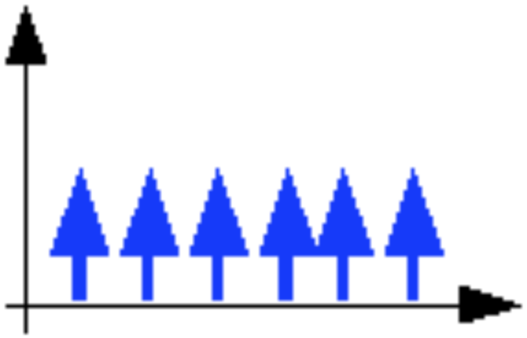
1. Erosion: $\phi(R_{jk}) = r_1$
   Take the minimum element in the ranking
2. Dilation: $\phi(R_{jk}) = r_Q$
   Take the maximum element in the ranking
3. Median: $\phi(R_{jk}) = r_{(Q+1)/2}$
   Take the middle element in the ranking

Examples of Different Operations

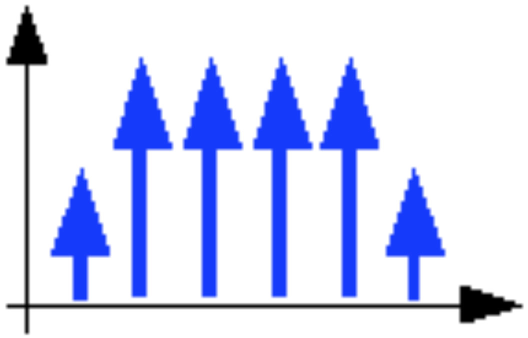


Original image

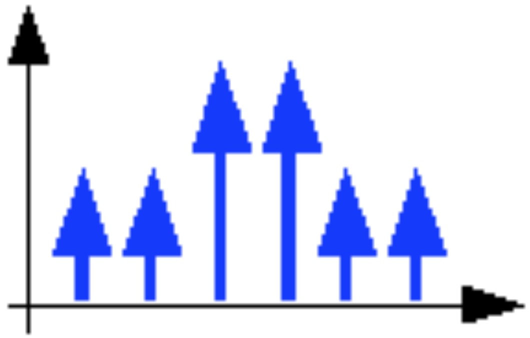
After mean filtering



After erosion



After dilation



After median filtering

Edge Detection via Morphology

- We can compute the edges in an image using dilation and erosion.

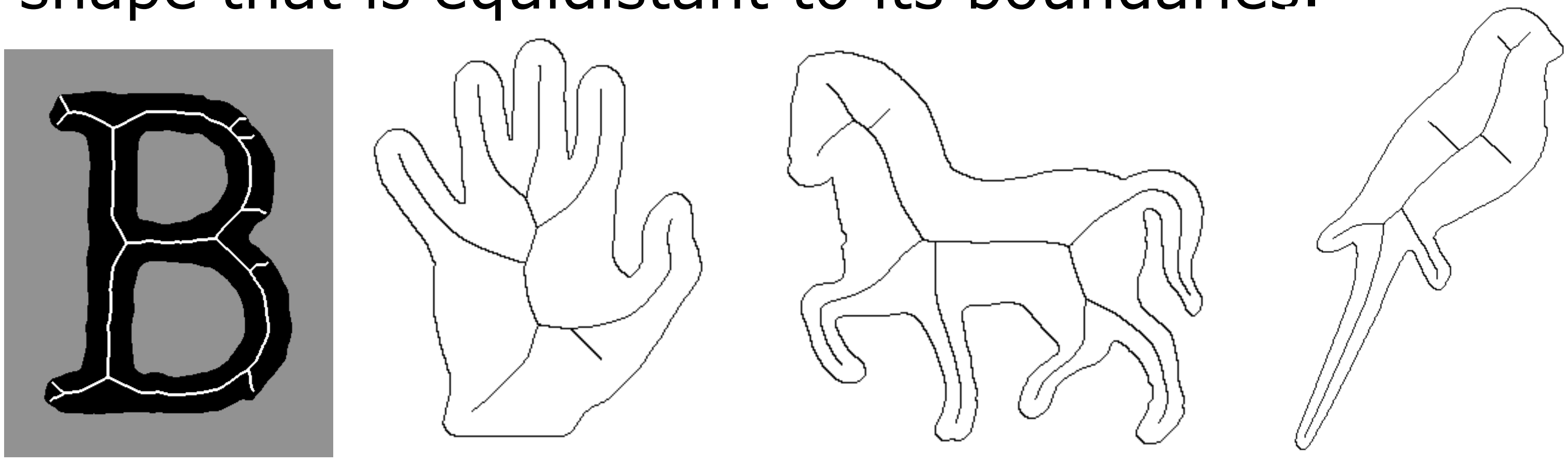
- Given a gray-scale image *f*:

1. Let f_d = dilation(*f*)
2. Let f_e = erosion(*f*)
3. Edge Image $E = f_d - f_e$

We can compute edges without ever approximating the gradient or any derivative (e.g. the Laplacian).

Skeleton

- In image analysis a skeleton (also known as medial axis transform) of a shape is a thin version of that shape that is equidistant to its boundaries.

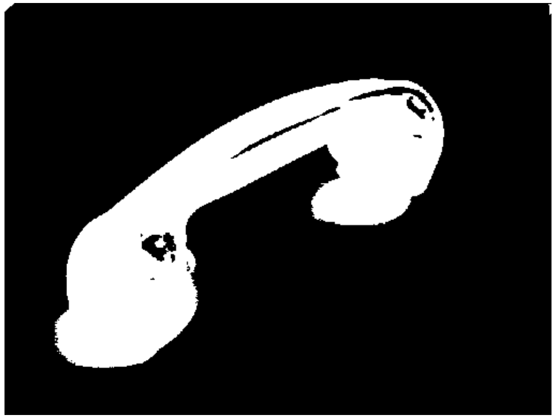


- One way of computing the skeleton is via repeated erosion followed by opening operations, or other similar combinations.

Images courtesy of wikipedia (letter B) and A. Telea and J.J. van Wijk.

Benefits of Closing

Original image



Binary image

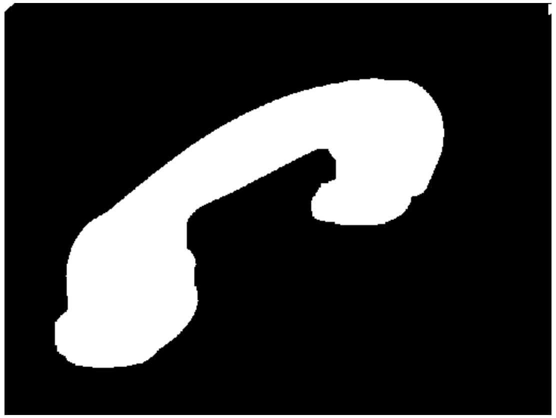
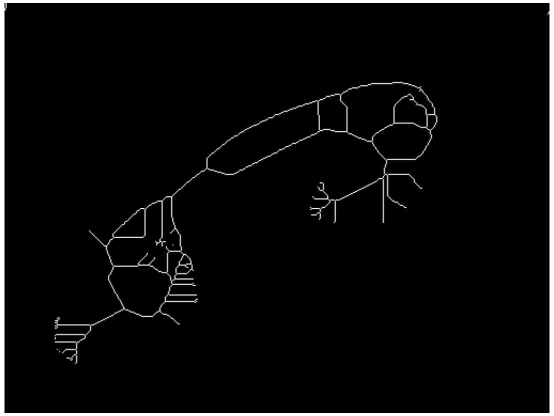
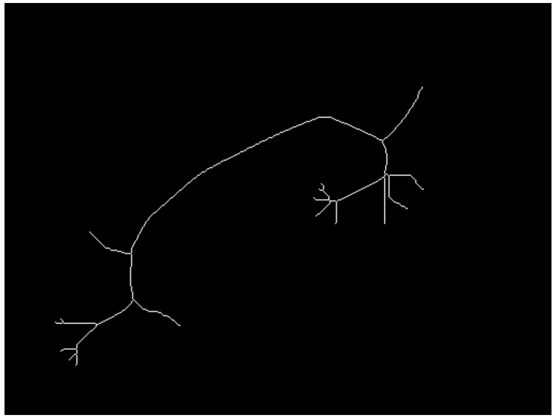


Image after closing



Skeleton without closing



Skeleton after closing

Images courtesy of R.Fisher, S. Perkins, W. Walker and E. Wolfart