The Daubechies construction (very short version)

- Goal: construct a filter

$$\boldsymbol{h} = (h_0, h_1, h_2, \ldots, h_L)$$

  which satisfies the $2M = L + 1$ orthogonality and low-pass conditions

- In terms of the z-transform

$$h(z) = h_0 + h_1 z + \ldots + h_L z^L$$

  one needs

  - for the low-pass conditions

$$h(z) = (1+z)^M \cdot q_M(z)$$

    for some real polynomial $q_{M-1}(z)$ of degree $M-1$

  - for the orthogonality conditions

$$|h(z)|^2 + |h(-z)|^2 = 2 \qquad \text{for } \; |z| = 1$$

- From these conditions one gets, by setting $z = e^{i\phi}$ and $y = \sin^2(\phi/2)$, the condition

$$(1-y)^M p_{M-1}(y) + y^M p_{M-1}(1-y) = 1$$

for some real polynomial $p_{M-1}(y)$ of degree $M-1$
- There are quite simple polynomials, which satisfy this condition:

$$P_M(y) = \sum_{k=0}^{M} \binom{M+k}{m} y^k$$

(the Daubechies polynomials)

- The first Daubechies polynomials

$$
\begin{aligned}
P_0 &= 1 \\
P_1 &= 1 + 2y \\
P_2 &= 1 + 3y + 6y^2 \\
P_3 &= 1 + 4y + 10y^2 + 20y^3 \\
P_4 &= 1 + 5y + 15y^2 + 35y^3 + 70y^4 \\
P_5 &= 1 + 6y + 21y^2 + 56y^3 + 126y^4 + 252y^5
\end{aligned}
$$

- To obtain the data for h
  1. take $P_{M-1}(y)$ and then
  $$\mathcal{P}_{2M-1}(y) = (1-y)^M P_{M-1}(y)$$
  2. in $\mathcal{P}_{2M-1}(y)$ make the substitution $y \to \frac{1}{2} - \frac{z+z^{-1}}{4}$ to get
  $$\mathrm{P}_{2M-1}(z) = \mathcal{P}_{2M-1}(y) = \sum_{k=-2M+1}^{2M-1} a_k z^k$$
  3. finally multiply by z^{2M-1} to obtain a polynomial
  $$\boldsymbol{P}_{4M-2}(z) = z^{2M-1}\mathrm{P}_{2M-1}(z)$$
  4. and this polynomial can be factored
  $$\boldsymbol{P}_{4M-2}(z) = (1+z)^{2M} Q_{2M-2}(z)$$
  5. indeed, the is a real polynomial $R_{M-1}(z)$ with
  $$\boldsymbol{P}_{4M-2}(z) = \left|(1+z)^M R_{M-1}(z)\right|^2$$

- the case $M = 1$

$$
\begin{aligned}
P_0(y) &= 1 \\
\mathcal{P}_1(y) &= 1 - y \\
\mathrm{P}_1(z) &= \frac{1}{4z} + \frac{1}{2} + \frac{z}{4} \\
\boldsymbol{P}_2(z) &= \frac{1}{4} + \frac{z}{2} + \frac{z^2}{4} \\
&= \frac{1}{4}(1 + z)^2
\end{aligned}
$$

- the case $M = 2$

$$\begin{aligned} P_1(y) &= 1 + 2y \\ \mathcal{P}_3(y) &= 1 - 3y^2 + 2y^3 \\ \mathrm{P}_3(z) &= \frac{1}{32}\left(-z^{-3} + 9z^{-1} + 16 + 9z - z^3\right) \\ \boldsymbol{P}_6(z) &= \frac{1}{32}\left(-1 + 9z^2 + 16z^3 + 9z^4 - z^6\right) \\ &= -\frac{1}{32}(z+1)^4\left(z^2 - 4z + 1\right) \end{aligned}$$

- the case $M = 3$

$$
\begin{aligned}
P_2(y) &= 1 + 3y + 6y^2 \\
\mathcal{P}_5(y) &= 1 - 10y^3 + 15y^4 - 6y^5 \\
\mathrm{P}_5(z) &= \frac{1}{512}\left(3z^{-5} - 25z^{-3} + 150z^{-1} + 256 + 150z - 25z^3 + 3z^5\right) \\
\boldsymbol{P}_{10}(z) &= \frac{1}{512}(z+1)^6\left(3z^4 - 18z^3 + 38z^2 - 18z + 3\right)
\end{aligned}
$$

- The case $M = 4$

$$\boldsymbol{P}_{14}(z) = -\frac{(z+1)^8\left(5z^6 - 40z^5 + 131z^4 - 208z^3 + 131z^2 - 40z + 5\right)}{4096}$$

- The case $M = 5$

$$\boldsymbol{P}_{14}(z) = \frac{(z+1)^{10}}{131072}\left(35z^8 - 350z^7 + 1520z^6 - 3650z^5 + 5018z^4 - 3650z^3 + 1520z^2 - 350z + 35\right)$$