Signals and Filters

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periodic functions (1)

- 2π-periodic functions can be identified with functions defined on the interval $I = [-\pi, \pi)$
- $\mathcal{L}^2(-\pi, \pi)$: (Hilbert-)space of square-integrable functions in I, i.e., functions $f : I \to \mathbb{C}$ with $\int_I |f(\omega)|^2 d\omega < \infty$ and (complex) inner product

$$\langle f \mid g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega)\, \overline{g(\omega)}\, d\omega$$

- $\mathcal{L}^2$-norm

$$\|f\|_2^2 = \langle f \mid f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\omega)|^2 d\omega.$$

periodic functions (2)

- The functions (for $k \in \mathbb{Z}$)

$$\varepsilon_k : \omega \mapsto e^{i\,k\,\omega}$$

  form a complete orthonormal basis (Hilbert-basis) of the space $\mathcal{L}^2(-\pi, \pi)$
- Proof of orthonormality:

$$\begin{aligned}\langle\, \varepsilon_j \,|\, \varepsilon_k \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\,j\,\omega} e^{-i\,k\,\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\,(j-k)\,\omega} d\omega \\ &= \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} 1\, d\omega = 1 & \text{if } j = k, \\ \frac{1}{2\pi i} \frac{1}{j-k}\, e^{i\,(j-k)\,\omega} \Big|_{-\pi}^{\pi} = 0 & \text{if } j \neq k. \end{cases}\end{aligned}$$

periodic functions (3)

- For any integrable 2π-periodic function f, i.e., $\int_I |f(\omega)|\, d\omega < \infty$, its *Fourier coefficients* are defined by

$$c_{f,k} = \langle\, f \mid \varepsilon_k \,\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega)\, e^{-i\,k\,\omega} d\omega$$

- For sufficiently well-behaved functions one has the expansion into a *Fourier series*:

$$f(\omega) \simeq \sum_{k \in \mathbb{Z}} c_{f,k}\, e^{i\,k\,\omega}$$

periodic functions (4)

- For functions $f, g \in \mathcal{L}^2(-\pi,\pi)$ the Parseval identity holds

$$\sum_{k\in\mathbb{Z}} c_{f,k} \cdot \overline{c_{g,k}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega)\cdot\overline{g(\omega)}\,d\omega = \langle\, f \mid g \,\rangle,$$

and so does the Plancherel identity

$$\sum_{k\in\mathbb{Z}} |c_{f,k}|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\omega)|^2\,d\omega = \|f\|_2^2$$

- *Proof*: It follows from orthonormality of the functions ε_k that

$$\begin{aligned}\langle\, f \mid g \,\rangle &= \langle \sum_j c_{f,j}\,\varepsilon_j \mid \sum_k c_{g,k}\,\varepsilon_k \,\rangle \\ &= \sum_j \sum_k c_{f,j}\,\overline{c_{g,k}}\,\langle\, \varepsilon_j \mid \varepsilon_k \,\rangle \\ &= \sum_k c_{j,k}\,\overline{c_{g,k}}\end{aligned}$$

signals (1)

- A (time-discrete) *signal* is a two-sided infinite sequence

$$\boldsymbol{x} = (\ldots, x[-2], x[-1], x[0], x[1], x[2|, \ldots) = (x[n])_{n\in\mathbb{Z}}$$

  of complex numbers, i.e., $\boldsymbol{x} \in \mathbb{C}^{\mathbb{Z}}$
- $\mathbb{C}^{\mathbb{Z}}$ is a $\mathbb{C}$-vector space (of uncountable dimension) w.r.t. component-wise addition and scalar multiplikation
- ℓ^1-signals are signals $\boldsymbol{x}$ with $\|\boldsymbol{x}\|_1 = \sum_n |x[n]| < \infty$
- ℓ^2-signals are signals $\boldsymbol{x}$ with $\|\boldsymbol{x}\|_2^2 = \sum_{n\in\mathbb{Z}} |x[n]|^2 < \infty$
- Every ℓ^1-signal is a ℓ^2-signal, but not conversely
- $\ell^1 = \ell^1(\mathbb{Z})$ resp. $\ell^2 = \ell^2(\mathbb{Z})$ denote the subspaces (with norm) of $\mathbb{C}^{\mathbb{Z}}$ of ℓ^1- resp. ℓ^2-signals. Both have countable dimension. ℓ^2 is even a Hilbert space w.r.t. the inner product

$$\langle\, \boldsymbol{x} \,, |\, \boldsymbol{y} \,\rangle = \sum_{n\in\mathbb{Z}} x[n] \cdot \overline{y[n]}$$

signals (2)

- The *frequency representation* of a signal $\boldsymbol{x}$ is its Fourier series

$$X(\omega) = \sum_{n \in \mathbb{Z}} x[n]\, e^{in\omega}$$

  This is a 2π-periodic function

- The coefficients are obtained from $X(\omega)$ by Fourier's integral:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{-i\,n\,\omega} d\omega = \langle\, X \,|\, \varepsilon_{\boldsymbol{n}} \,\rangle$$

signals (3)

- For ℓ^2-signals one has energy conservation (see the Plancherel formula above)

$$\|\boldsymbol{x}\|_2^2 = \sum_{n\in\mathbb{Z}} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega = \|X(\omega)\|_2^2,$$

- and more generally the Parseval identity holds:

$$\langle\, \boldsymbol{x} \mid \boldsymbol{y}\,\rangle = \sum_{n\in\mathbb{Z}} x[n]\,\overline{y[n]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\overline{Y}(\omega)\, d\omega = \langle\, X(\omega) \mid Y(\omega)\,\rangle$$

signals (4)

- The *unit impulse* at time 0 is the signal $\boldsymbol{\delta}$ given by

$$\boldsymbol{\delta}[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

- For any signal $\boldsymbol{x}$ and $k \in \mathbb{Z}$ the *k-shifted* signal $\tau^k \boldsymbol{x}$ (by *time* or *distance* k) is given by

$$\left(\tau^k \boldsymbol{x}\right)[n] = x[n-k] \quad (n \in \mathbb{Z})$$

- The linear mappings

$$\tau^k : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}} : \boldsymbol{x} \mapsto \tau^k \boldsymbol{x}$$

are also linear transformations of ℓ^1 and of ℓ^2

convolution (1)

- The *convolution* $\boldsymbol{x} \star \boldsymbol{y}$ of two signals $\boldsymbol{x} = (x[n])_{n\in\mathbb{Z}}$ and $\boldsymbol{y} = (y[n])_{n\in\mathbb{Z}}$ is defined by

$$(\boldsymbol{x} \star \boldsymbol{y})[n] = \sum_{k\in\mathbb{Z}} x[k] \cdot y[n-k] \qquad (n \in \mathbb{N})$$

(provided that the sums converge for all $n \in \mathbb{Z}$)

- Convolution is *commutative* and *associative*:

$$\boldsymbol{x} \star \boldsymbol{y} = \boldsymbol{y} \star \boldsymbol{x} \quad \text{and} \quad \boldsymbol{x} \star (\boldsymbol{y} \star \boldsymbol{z}) = (\boldsymbol{x} \star \boldsymbol{y}) \star \boldsymbol{z}$$

convolution (2)

- The most important special cases are these:
  - If $\boldsymbol{x} = (x[n])_{n \in \mathbb{Z}}$ is a *finite* signal (finitely many $x[n]$ are $\neq 0$), then
    – for any signal $\boldsymbol{y}$ the convolution $\boldsymbol{x} \star \boldsymbol{y}$ is again a signal, and
    – for $\boldsymbol{y} \in \ell^1$ resp. $\in \ell^2$ the conv. $\boldsymbol{x} \star \boldsymbol{y}$ is again $\in \ell^1$ resp. $\in \ell^2$.
    The same is true if $\boldsymbol{y}$ is a finite signal.
  - For $\boldsymbol{x}, \boldsymbol{y} \in \ell^1$, one has $\boldsymbol{x} \star \boldsymbol{y} \in \ell^1$. This follows from

$$
\begin{aligned}
\|\mathbf{x} \star \mathbf{y}\|_1 &= \sum_{n \in \mathbb{Z}} |(\boldsymbol{x} \star \boldsymbol{y})[k]| \\
&= \sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} x[k] \cdot y[n-k] \right| \\
&\leq \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |x[k]| \cdot |y[n-k]| \\
&= \sum_{k \in \mathbb{Z}} |x[k]| \cdot \sum_{n \in \mathbb{Z}} |y[n]| = \|\boldsymbol{x}\|_1 \cdot \|\boldsymbol{y}\|_1
\end{aligned}
$$

  - For ℓ^2 the correct statement is a bit more complicated

convolution (3)

- *The convolution theorem*:
  For signals $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \ell^1$ with $\boldsymbol{z} = \boldsymbol{x} \star \boldsymbol{y}$ one has

  $$\forall \omega : Z(\omega) = X(\omega) \cdot Y(\omega)$$

  for the corresponding Fourier series
- This follows from

$$\begin{aligned} Z(\omega) &= \sum_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} x[k]\, y[n-k] \right) e^{i\,n\,\omega} \\ &= \sum_{n,k \in \mathbb{Z}} x[k]\, e^{i\,k\,\omega}\, y[n-k]\, e^{i\,(n-k)\,\omega} \\ &= \sum_{k \in \mathbb{Z}} x[k]\, e^{i\,k\,\omega} \cdot \sum_{n \in \mathbb{Z}} y[n]\, e^{i\,n\,\omega} \\ &= X(\omega) \cdot Y(\omega) \end{aligned}$$

filter (1)

- A linear transformation $T : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ (or of ℓ^1 resp. ℓ^2) is *translation invariant*, if it commutes with the shift τ:

$$\forall \boldsymbol{x} \in \mathbb{C}^{\mathbb{Z}} : T(\tau \boldsymbol{x}) = \tau(T\boldsymbol{x}),$$

  in shorthand: $T \circ \tau = \tau \circ T$.
- If this holds, then $T \circ \tau^k = \tau^k \circ T$ for all $k \in \mathbb{Z}$

filter (2)

- A linear transformation $T : \ell^1 \to \ell^1$ is *continuous* (or *stable*), if there is a constant $C > 0$ such that

$$\forall \boldsymbol{x} \in \ell^1 : \|T\boldsymbol{x}\|_1 \leq C \cdot \|\boldsymbol{x}\|_1$$

- An equivalent statement is:
for every sequence of signals $\left(\boldsymbol{x}^{(m)}\right)_{m\in\mathbb{N}}$ in ℓ^1 and $\boldsymbol{x} \in \ell^1$ with $\left(\boldsymbol{x}^{(m)}\right)_{m\in\mathbb{N}} \to_{\ell^1} \boldsymbol{x}$ one has $\left(T\boldsymbol{x}^{(m)}\right)_{m\in\mathbb{N}} \to_{\ell^1} T\boldsymbol{x}$, i.e.,

$$\|\boldsymbol{x}^{(m)} - \boldsymbol{x}\| \to_{m\to\infty} 0 \quad \Rightarrow \quad \|T\boldsymbol{x}^{(m)} - T\boldsymbol{x}\| \to_{m\to\infty} 0$$

- A similar definition is made for transformations of ℓ^2

Filter (3)

- Definition: An ℓ^1- *filter* resp. ℓ^2-*filter* is a linear transformation of ℓ^1 resp. ℓ^2 which is both translation invariant and continuous
- For any $\boldsymbol{h} \in \ell^1$ the convolution mapping

$$T_{\boldsymbol{h}} : \ell^1 \to \ell^1 : \boldsymbol{x} \mapsto \boldsymbol{x} \star \boldsymbol{h}$$

  is an ℓ^1-Filter
  - translation invariance can be checked directly
  - continuity follows from

$$\|T_{\boldsymbol{h}}\boldsymbol{x}\|_1 = \|\boldsymbol{x} \star \boldsymbol{h}\| \leq \|\boldsymbol{x}\|_1 \cdot \|\boldsymbol{h}\|_1$$

    The required constant C is just $\|\boldsymbol{h}\|_1$

filter (4)

- Theorem: For any ℓ^1-Filter T there is an $\boldsymbol{h} \in \ell^1$ s.th. $T = T_{\boldsymbol{h}}$.
- Sketch of proof:
  - Write the signal $\boldsymbol{x}$ as a linear combination of shifted impulses:

$$\boldsymbol{x} = \sum_{k \in \mathbb{Z}} x[k]\, \tau^k \boldsymbol{\delta}$$

  - Now put $\boldsymbol{h} = T\boldsymbol{\delta}$.
    It follows from linearity and translation invariance of T that

$$T\boldsymbol{x} \stackrel{(*)}{=} \sum_k x[k]\, T\tau^k \boldsymbol{\delta} = \sum_k x[k]\, \tau^k T\boldsymbol{\delta} = \sum_k x[k]\, \tau^k \boldsymbol{h}$$

  - From $(\tau^k \boldsymbol{h})[n] = h[n-k]$ one has

$$(T\boldsymbol{x})[n] = \sum_{k \in \mathbb{Z}} x[k]\, h[n-k].$$

    and thus $T\boldsymbol{x} = \boldsymbol{x} \star \boldsymbol{h}$
  - Notabene: Continuity of T is needed in order to justify switching of T with the infinite sum $\sum_k$ in $(*)$

filter (5)

- About teminology: the signal $\boldsymbol{h} = T\boldsymbol{\delta}$ is called *impulse response* of the filter. The corresponding Fourier series $H(\omega)$ is the *frequency response* or *transfer function* of the filter
- In systems theory, the *z-transform* of a signal (or filter) $\boldsymbol{h} = (h[k])_{k\in\mathbb{Z}}$ is the power series

  $$h(z) = \sum_{k\in\mathbb{Z}} h[k]\, z^k,$$

  so that the frequency response is $H(\omega) = h(e^{i\,\omega})$
- Writing $H(\omega)$ for real ω is the same as considering $h(z)$ only for z from the complex unit circle, i.e. $|z| = 1$. In writing $h(z)$ one implicitly considers z as a general complex variable
- Some authors define $H(\omega) = h(e^{-i\,\omega})$, in which case $H(\omega) = \sum_k h[k]\, e^{-i\,k\,\omega}$

filter (6)

- The "harmonic" signal $\boldsymbol{x}_\omega = \left(e^{-i\,n\,\omega}\right)_{n\in\mathbb{Z}}$ belongs neither to ℓ^1 nor to ℓ^2, but the convolution $T_{\boldsymbol{h}}\,\boldsymbol{x}_\omega = \boldsymbol{x}_\omega \star \boldsymbol{h}$ can be computed for any $\boldsymbol{h} \in \ell^1$:

$$\begin{aligned}(\boldsymbol{x}_\omega * \boldsymbol{h})[n] &= \sum_{k\in\mathbb{Z}} e^{-i\,k\,\omega} h[n-k] \\ &= e^{-i\,n\,\omega} \sum_{k\in\mathbb{Z}} e^{i(n-k)\omega} h[n-k] = H(\omega)\cdot e^{-i\,n\,\omega}\end{aligned}$$

or $\quad T_{\boldsymbol{h}}\,\boldsymbol{x}_\omega = H(\omega)\cdot \boldsymbol{x}_\omega$

- This means: each harmonic $\boldsymbol{x}_\omega = \left(e^{-i\,n\,\omega}\right)_{n\in\mathbb{Z}}$ is an eigenvector of $T_{\boldsymbol{h}}$ with eigenvalue $H(\omega)$
- Conclusion: If $T = T_{\boldsymbol{h}}$ is an ℓ^1-filter with frequency response $H(\omega)$, then for any ℓ^1-signal $\boldsymbol{x}$ and $\boldsymbol{y} = T\boldsymbol{x} = \boldsymbol{x} \star \boldsymbol{h}$ one has

$$\forall\omega : Y(\omega) = X(\omega)\cdot H(\omega)$$

filter (7)

- The corresponding ℓ^2-theory is technically a bit more complicated, but the results are essentially the same:

  ℓ^2-filters are precisely the convolution transformations

  $$T_{\boldsymbol{h}} : \boldsymbol{x} \mapsto \boldsymbol{x} \star \boldsymbol{h},$$

  for which $H(\omega) \in \mathcal{L}^\infty(-\pi, \pi)$, i.e. $H(\omega)$ is bounded.

filter (8)

- A filter $T = T_{\boldsymbol{h}}$ is *real*, if $h[n] \in \mathbb{R}$ for all $n \in \mathbb{Z}$
- For a real filter $\boldsymbol{h}$ one has

$$\overline{H(\omega)} = \sum_n h[n] e^{-i\,n\,\omega} = H(-\omega)$$

- Consequently $|H(\omega)| = |H(-\omega)|$, i.e., the function $\omega \mapsto |H(\omega)|$ is an *even* function. It suffices to know this function on the interval $[0, \pi]$

filter (9)

- A filter $T = T_{\boldsymbol{h}}$ is *causal*, if $h[n] = 0$ for all $n < 0$
- For a causal filter $\boldsymbol{h}$ one has for $\boldsymbol{y} = T_{\boldsymbol{h}}\boldsymbol{x}$:

$$y[n] = \sum_{k \leq n} x[k]\, h[n-k] = \sum_{k \geq 0} x[n-k]\, h[k],$$

  i.e., the response (output) $y[n]$ at time n only depends on the inputs $x[n-k]$ at previous times $n - k \leq n$

filter (10)

- A filter $T = T_{\boldsymbol{h}}$ is a *FIR-filter* (*finite impulse response*), if $h[n] \neq 0$ only for a finite number of filter coefficients

- A FIR-Filter is specified by a finite vector of filter coefficients $(h[a], h[a+1], \ldots, h[b])$ with $a < b$ and $h[a] \neq 0 \neq h[b]$

downsampling und upsampling (1)

- For any signal $\boldsymbol{x}$ one denotes by $\boldsymbol{y} = \downarrow_2 \boldsymbol{x}$ (2-*downsampling*) the signal given by

$$y[n] = x[2n] \quad (n \in \mathbb{Z})$$

(coefficients with odd index are eliminated)

- This is not a filtering operation because it is not translation-invariant! In general:

$$(\downarrow_2 \tau^k \boldsymbol{x})[0] = x[-k] \neq x[-2k] = (\tau^k \downarrow_2 \boldsymbol{x})[0]$$

- As for the frequency representation, one has (because of $(-1)^n = e^{i\,n\,\pi}$):

$$Y(\omega) = \sum_{n\in\mathbb{Z}} x[2n]\, e^{i\,n\,\omega} = \sum_{n\in\mathbb{Z}} x[n]\, \frac{1+(-1)^n}{2}\, e^{i\,n\,\omega/2}$$

$$= \frac{1}{2}\Big(X(\frac{\omega}{2}) + X(\frac{\omega}{2} + \pi)\Big)$$

downsampling und upsampling (2)

- For any signal $\boldsymbol{x}$ one denotes by $\boldsymbol{y} = \uparrow_2 \boldsymbol{x}$ (2-*upsampling*) the signal given by

$$y[n] = \begin{cases} x[n/2] & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

  (inserting 0 between any two neighboring coefficients of $\boldsymbol{x}$)
- This is not a filtering operation because it is not translation-invariant!
- As for the frequency representation, one has

$$Y(\omega) = \sum_{n \in \mathbb{Z}} x[n] e^{i\,2n\,\omega} = X(2\omega)$$

downsampling und upsampling (3)

- Downsampling and upsampling do not commute!

  One has $\downarrow_2 \uparrow_2 \boldsymbol{x} = \boldsymbol{x}$, but for $\boldsymbol{y} = \uparrow_2 \downarrow_2 \boldsymbol{x}$ one gets

$$y[n] = \begin{cases} x[n] & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

  with frequency representation

$$Y(\omega) = \frac{1}{2}\big(X(\omega) + X(\omega + \pi)\big)$$