

Discrete wavelet transform (Synopsis)

- Filters

$$h = (h_0, h_1, \dots, h_L) \quad \text{with} \quad h(z) = h_0 + h_1(z) + \dots + h_L z^L \quad \text{and} \quad H(\omega) = h(e^{i\omega})$$

$$g = (g_0, g_1, \dots, g_L) \quad \text{with} \quad g(z) = g_0 + g_1(z) + \dots + g_L z^L \quad \text{and} \quad G(\omega) = g(e^{i\omega})$$

related by

$$g_k = (-1)^k h_{L-k} \quad (0 \leq k \leq L), \quad \text{i.e.,} \quad g(z) = (-z)^L h\left(-\frac{1}{z}\right)$$

- Orthogonality conditions

$$\begin{aligned} h(z) \cdot h\left(\frac{1}{z}\right) + h(-z) \cdot h\left(-\frac{1}{z}\right) &= 2 & |H(\omega)|^2 + |H(\omega + \pi)|^2 &= 2 \\ g(z) \cdot g\left(\frac{1}{z}\right) + g(-z) \cdot g\left(-\frac{1}{z}\right) &= 2 & |G(\omega)|^2 + |H(\omega + \pi)|^2 &= 2 \\ h(z) \cdot g\left(\frac{1}{z}\right) + h(-z) \cdot g\left(-\frac{1}{z}\right) &= 0 & H(\omega) \cdot \overline{G(\omega)} + H(\omega + \pi) \cdot \overline{G(\omega + \pi)} &= 0 \end{aligned}$$

- Low-/high-pass conditions

$$\begin{aligned} H(0) &= \sqrt{2} & G(\pi) &= \sqrt{2} \\ H^{(j)}(\pi) &= 0 & G^{(j)}(0) &= 0 & j &= 0, 1, 2, \dots \\ H^{(j)}(0) &= 0 & G^{(j)}(\pi) &= 0 & j &= 1, 2, \dots \end{aligned}$$

- Analysis transformation

$$H : a(z) \mapsto \frac{1}{2} [h(z) \cdot a(z) + h(-z) \cdot a(-z)]_{z^2 \leftarrow z}$$

$$G : a(z) \mapsto \frac{1}{2} [g(z) \cdot a(z) + g(-z) \cdot a(-z)]_{z^2 \leftarrow z}$$

- Synthesis transformation

$$H^\dagger : a(z) \mapsto h\left(\frac{1}{z}\right) \cdot a(z^2) \quad G^\dagger : a(z) \mapsto g\left(\frac{1}{z}\right) \cdot a(z^2)$$

- Reconstruction condition

$$\begin{aligned}
H^\dagger \cdot H + G^\dagger \cdot G &= I \\
h(z) \cdot h\left(\frac{1}{z}\right) + g(z) \cdot g\left(\frac{1}{z}\right) &= 2 \\
h(z) \cdot h\left(-\frac{1}{z}\right) + g(z) \cdot g\left(-\frac{1}{z}\right) &= 0
\end{aligned}$$

- Transformation of signals of finite length N

$$\begin{aligned}
H_N = H_{N,1} &= \begin{bmatrix} h_L & h_{L-1} & h_{L-2} & h_{L-3} & \dots & h_1 & h_0 & 0 & \dots & 0 \\ 0 & 0 & h_L & h_{L-1} & \dots & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{L-2} & h_{L-3} & \dots & \dots & \dots & \dots & \dots & & h_L & h_{L-1} \end{bmatrix}_{N/2 \times N} \\
G_N = G_{N,1} &= \begin{bmatrix} g_L & g_{L-1} & g_{L-2} & g_{L-3} & \dots & g_1 & g_0 & 0 & \dots & 0 \\ 0 & 0 & g_L & g_{L-1} & \dots & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{L-2} & g_{L-3} & \dots & \dots & \dots & \dots & \dots & & g_L & g_{L-1} \end{bmatrix}_{N/2 \times N} \\
W_N = W_{N,1} &= \begin{bmatrix} H_{N,1} \\ G_{N,1} \end{bmatrix}_{N \times N}
\end{aligned}$$

- One-level analysis transform

$$W_{N,1} : \mathbf{x} \mapsto \begin{bmatrix} H_{N,1} \mathbf{x} \\ G_{N,1} \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{d}^{(1)} \end{bmatrix}$$

- One-level synthesis transform

$$W_{N,1}^\dagger : \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{d}^{(1)} \end{bmatrix} \mapsto \begin{bmatrix} H_{N,1}^\dagger & G_{N,1}^\dagger \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{d}^{(1)} \end{bmatrix} = \underbrace{H_{N,1}^\dagger H_{N,1} \mathbf{x}}_{\mathbf{x}^{(1)}} + \underbrace{G_{N,1}^\dagger G_{N,1} \mathbf{x}}_{\mathbf{y}^{(1)}}$$

- Projections and coefficients

$\mathbf{x}^{(1)} = P_{N,1} \mathbf{x} = H_{N,1}^\dagger H_{N,1} \mathbf{x}$: projection onto the approximation space

$\mathbf{y}^{(1)} = Q_{N,1} \mathbf{x} = G_{N,1}^\dagger G_{N,1} \mathbf{x}$: projection onto the detail space

$\mathbf{a}^{(1)} = H_{N,1} \mathbf{x}$: approximation coefficients

$\mathbf{d}^{(1)} = G_{N,1} \mathbf{x}$: detail coefficients

Properties

$$P_{N,1}^2 = P_{N,1}, \quad Q_{N,1}^2 = Q_{N,1}, \quad P_{N,1}Q_{N,1} = Q_{N,1}P_{N,1} = 0, \quad P_{N,1} + Q_{N,1} = I$$

– Multilevel transform

* Matrices

$$H_{N,k} = H_{N/2^{k-1},1} \cdot H_{N,k-1} = H_{N/2^{k-1},1} H_{N/2^{k-2},1} \cdots H_{N/2,1} H_{N,1}$$

$$G_{N,k} = G_{N/2^{k-1},1} \cdot H_{N,k-1} = G_{N/2^{k-1},1} H_{N/2^{k-2},1} \cdots H_{N/2,1} H_{N,1}$$

$$W_{N,k}^t = \begin{bmatrix} H_{N,k}^t & G_{N,k}^t & G_{N,k-1}^t & \cdots & G_{N,1}^t \end{bmatrix}_{N \times N}$$

* Analysis (approximation and detail coefficients)

$$W_{n,k} : \mathbf{x} \mapsto \begin{bmatrix} \mathbf{a}^{(k)} \\ \mathbf{d}^{(k)} \\ \mathbf{d}^{(k-1)} \\ \vdots \\ \mathbf{d}^{(1)} \end{bmatrix} = \begin{bmatrix} H_{N,k} \mathbf{x} \\ G_{N,k} \mathbf{x} \\ G_{N,k-1} \mathbf{x} \\ \vdots \\ G_{N,1} \mathbf{x} \end{bmatrix}_{N \times N}$$

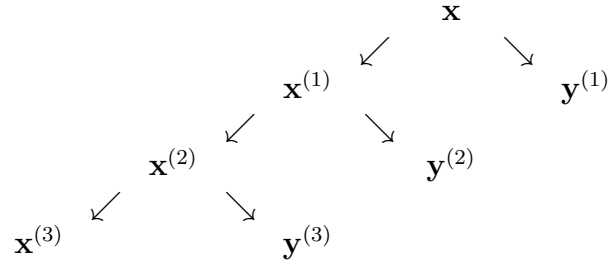
* Synthesis

$$\begin{aligned} W_{n,k}^\dagger : \begin{bmatrix} \mathbf{a}^{(k)} \\ \mathbf{d}^{(k)} \\ \mathbf{d}^{(k-1)} \\ \vdots \\ \mathbf{d}^{(1)} \end{bmatrix} &\mapsto \begin{bmatrix} H_{N,k}^\dagger & G_{N,k}^\dagger & G_{N,k-1}^\dagger & \cdots & G_{N,1}^\dagger \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(k)} \\ \mathbf{d}^{(k)} \\ \mathbf{d}^{(k-1)} \\ \vdots \\ \mathbf{d}^{(1)} \end{bmatrix} \\ &= \underbrace{H_{N,k}^\dagger H_{N,k}}_{\mathbf{x}^{(k)}} \mathbf{x} + \underbrace{G_{N,k}^\dagger G_{N,k}}_{\mathbf{y}^{(k)}} \mathbf{x} + \underbrace{G_{N,k-1}^\dagger G_{N,k-1}}_{\mathbf{y}^{(k-1)}} \mathbf{x} + \cdots + \underbrace{G_{N,1}^\dagger G_{N,1}}_{\mathbf{y}^{(1)}} \mathbf{x} \end{aligned}$$

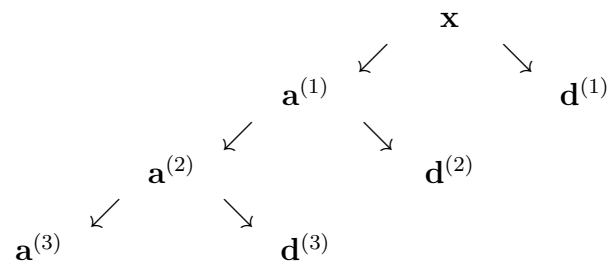
* $P_{n,k}$ and $Q_{n,k}$ are projections onto the corresponding approximation resp. detail spaces.

$\mathbf{x}^{(k)}$ and $\mathbf{y}^{(k)}$ are the images under these projections

- Projection scheme (for $k = 3$)



- Coefficient scheme (for $k = 3$)



- Note:

- The projection matrices $P_{N,k}$ and $Q_{N,k}$ are $N \times N$ matrices
- The images $\mathbf{x}^{(k)}$ resp. $\mathbf{y}^{(k)}$ under the projections onto the approximation and detail spaces are vectors of length N
- The lengths of the coefficient vectors $\mathbf{a}^{(k)}$ and $\mathbf{d}^{(k)}$ are $N/2^k$.
- The transformation given by

$$\mathbf{x} \mapsto [\mathbf{a}^{(k)} \quad \mathbf{d}^{(k)} \quad \mathbf{d}^{(k-1)} \quad \dots \quad d^{(1)}]^\top$$

is the k -level wavelet transform

- The representation

$$\mathbf{x} = \mathbf{x}^{(k)} + \mathbf{y}^{(k)} + \mathbf{y}^{(k-1)} + \dots + \mathbf{y}^{(1)}$$

is the decomposition of the signal \mathbf{x} using projections onto the approximation and wavelet spaces