Discrete wavelet transform (Synopsis)

- Filters

$$h = (h_0, h_1, \dots, h_L) \quad \text{with} \quad h(z) = h_0 + h_1(z) + \cdots + h_L z^L \quad \text{and} \quad H(\omega) = h(e^{i\omega})$$
$$g = (g_0, g_1, \dots, g_L) \quad \text{with} \quad g(z) = g_0 + g_1(z) + \cdots + g_L z^L \quad \text{and} \quad G(\omega) = g(e^{i\omega})$$

related by

$$g_k = (-1)^k h_{L-k} \quad (0 \le k \le L), \quad i.e., \quad g(z) = (-z)^L h(-\frac{1}{z}))$$

- Orthogonality conditions

$$h(z) \cdot h(\frac{1}{z}) + h(-z) \cdot h(-\frac{1}{z}) = 2 \qquad |H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$
$$g(z) \cdot g(\frac{1}{z}) + g(-z) \cdot g(-\frac{1}{z}) = 2 \qquad |G(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$
$$h(z) \cdot g(\frac{1}{z}) + h(-z) \cdot g(-\frac{1}{z}) = 0 \qquad H(\omega) \cdot \overline{G(\omega)} + H(\omega + \pi) \cdot \overline{G(\omega + \pi)} = 0$$

- Low-/high-pass conditions

$$H(0) = \sqrt{2} \qquad G(\pi) = \sqrt{2}$$
$$H^{(j)}(\pi) = 0 \qquad G^{(j)}(0) = 0 \qquad j = 0, 1, 2, \dots$$
$$H^{(j)}(0) = 0 \qquad G^{(j)}(\pi) = 0 \qquad j = 1, 2, \dots$$

- Analysis transformation

$$H : a(z) \mapsto \frac{1}{2} \left[h(z) \cdot a(z) + h(-z) \cdot a(-z)\right]_{z^2 \leftarrow z}$$
$$G : a(z) \mapsto \frac{1}{2} \left[g(z) \cdot a(z) + g(-z) \cdot a(-z)\right]_{z^2 \leftarrow z}$$

- Synthesis transformation

$$H^\dagger : a(z) \mapsto h(\frac{1}{z}) \cdot a(z^2) \qquad G^\dagger : a(z) \mapsto g(\frac{1}{z}) \cdot a(z^2)$$

- Reconstruction condition

$$H^\dagger \cdot H + G^\dagger \cdot G = I$$
$$h(z) \cdot h(\frac{1}{z}) + g(z) \cdot g(\frac{1}{z}) = 2$$
$$h(z) \cdot h(-\frac{1}{z}) + g(z) \cdot g(-\frac{1}{z}) = 0$$

- Transformation of signals of finite length N

$$H_N = H_{N,1} = \begin{bmatrix} h_L & h_{L-1} & h_{L-2} & h_{L-3} & \dots & h_1 & h_0 & 0 & \dots & 0 \\ 0 & 0 & h_L & h_{L-1} & \dots & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{L-2} & h_{L-3} & \dots & \dots & \dots & \dots & \dots & & h_L & h_{L-1} \end{bmatrix}_{N/2 \times N}$$

$$G_N = G_{N,1} = \begin{bmatrix} g_L & g_{L-1} & g_{L-2} & g_{L-3} & \dots & g_1 & g_0 & 0 & \dots & 0 \\ 0 & 0 & g_L & g_{L-1} & \dots & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{L-2} & g_{L-3} & \dots & \dots & \dots & \dots & \dots & & g_L & g_{L-1} \end{bmatrix}_{N/2 \times N}$$

$$W_N = W_{N,1} = \begin{bmatrix} H_{N,1} \\ G_{N,1} \end{bmatrix}_{N \times N}$$

  - One-level analysis transform

$$W_{N,1} : \mathbf{x} \longmapsto \begin{bmatrix} H_{N,1}\mathbf{x} \\ G_{N,1}\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{d}^{(1)} \end{bmatrix}$$

  - One-level synthesis transform

$$W_{N,1}^\dagger : \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{d}^{(1)} \end{bmatrix} \longmapsto \begin{bmatrix} H_{N,1}^\dagger G_{N,1}^\dagger \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{d}^{(1)} \end{bmatrix} = \underbrace{\overbrace{H_{N,1}^\dagger H_{N,1}}^{P_{N,1}} \mathbf{x}}_{\mathbf{x}^{(1)}} + \underbrace{\overbrace{G_{N,1}^\dagger G_{N,1}}^{Q_{N,1}} \mathbf{x}}_{\mathbf{y}^{(1)}}$$

  - Projections and coefficients

$$\mathbf{x}^{(1)} = P_{N,1}\mathbf{x} = H_{N,1}^\dagger H_{N,1}\mathbf{x} : \text{projection onto the approximation space}$$
$$\mathbf{y}^{(1)} = Q_{N,1}\mathbf{x} = G_{N,1}^\dagger G_{N,1}\mathbf{x} : \text{projection onto the detail space}$$
$$\mathbf{a}^{(1)} = H_{N,1}\mathbf{x} : \text{approximation coefficients}$$
$$\mathbf{d}^{(1)} = G_{N,1}\mathbf{x} : \text{detail coefficients}$$

Properties

$$P_{N,1}^2 = P_{N,1},\ \ Q_{N,1}^2 = Q_{N,1},\ \ P_{N,1}Q_{N,1} = Q_{N,1}P_{N,1} = 0,\ \ P_{N,1}+Q_{N,1} = I$$

– Multilevel transform

* Matrices

$$\begin{aligned} H_{N,k} &= H_{N/2^{k-1},1} \cdot H_{N,k-1} = H_{N/2^{k-1},1}H_{N/2^{k-2},1} \cdots H_{N/2,1}H_{N,1} \\ G_{N,k} &= G_{N/2^{k-1},1} \cdot H_{N,k-1} = G_{N/2^{k-1},1}H_{N/2^{k-2},1} \cdots H_{N/2,1}H_{N,1} \\ W_{N,k}^t &= \begin{bmatrix} H_{N,k}^t & G_{N,k}^t & G_{N,k-1}^t & \vdots & G_{N,1}^t \end{bmatrix}_{N\times N} \end{aligned}$$

* Analysis (approximation and detail coefficients)

$$W_{n,k} : \mathbf{x} \longmapsto \begin{bmatrix} \mathbf{a}^{(k)} \\ \mathbf{d}^{(k)} \\ \mathbf{d}^{(k-1)} \\ \vdots \\ \mathbf{d}^{(1)} \end{bmatrix} = \begin{bmatrix} H_{N,k}\mathbf{x} \\ G_{N,k}\mathbf{x} \\ G_{N,k-1}\mathbf{x} \\ \vdots \\ G_{N,1}\mathbf{x} \end{bmatrix}_{N\times N}$$

* Synthesis

$$W_{n,k}^\dagger : \begin{bmatrix} \mathbf{a}^{(k)} \\ \mathbf{d}^{(k)} \\ \mathbf{d}^{(k-1)} \\ \vdots \\ \mathbf{d}^{(1)} \end{bmatrix} \longmapsto \begin{bmatrix} H_{N,k}^\dagger & G_{N,k}^\dagger & G_{N,k-1}^\dagger & \dots & G_{N,1}^\dagger \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(k)} \\ \mathbf{d}^{(k)} \\ \mathbf{d}^{(k-1)} \\ \vdots \\ \mathbf{d}^{(1)} \end{bmatrix}$$

$$= \underbrace{\overbrace{H_{N,k}^\dagger H_{N,k}}^{P_{N,k}}\mathbf{x}}_{\mathbf{x}^{(k)}} + \underbrace{\overbrace{G_{N,k}^\dagger G_{N,k}}^{Q_{N,k}}\mathbf{x}}_{\mathbf{y}^{(k)}} + \underbrace{\overbrace{G_{N,k-1}^\dagger G_{N,k-1}}^{Q_{N,k-1}}\mathbf{x}}_{\mathbf{y}^{(k-1)}} + \cdots + \underbrace{\overbrace{G_{N,1}^\dagger G_{N,1}}^{Q_{N,1}}\mathbf{x}}_{\mathbf{y}^{(1)}}$$

* $P_{n,k}$ and $Q_{n,k}$ are projections onto the corresponding approximation resp. detail spaces.
$\mathbf{x}^{(k)}$ and $\mathbf{y}^{(k)}$ are the images under these projections

- Projection scheme (for $k = 3$)

$$
\begin{array}{ccccccc}
 & & & \mathbf{x} & & & \\
 & & \swarrow & & \searrow & & \\
 & & \mathbf{x}^{(1)} & & & & \mathbf{y}^{(1)} \\
 & \swarrow & & \searrow & & & \\
 & \mathbf{x}^{(2)} & & & \mathbf{y}^{(2)} & & \\
 \swarrow & & \searrow & & & & \\
 \mathbf{x}^{(3)} & & \mathbf{y}^{(3)} & & & &
\end{array}
$$

- Coefficient scheme (for $k = 3$)

$$
\begin{array}{ccccccc}
 & & & \mathbf{x} & & & \\
 & & \swarrow & & \searrow & & \\
 & & \mathbf{a}^{(1)} & & & & \mathbf{d}^{(1)} \\
 & \swarrow & & \searrow & & & \\
 & \mathbf{a}^{(2)} & & & \mathbf{d}^{(2)} & & \\
 \swarrow & & \searrow & & & & \\
 \mathbf{a}^{(3)} & & \mathbf{d}^{(3)} & & & &
\end{array}
$$

- Note:
  - The projection matrices $P_{N,k}$ and $Q_{N,k}$ are $N \times N$ matrices
  - The images $\mathbf{x}^{(k)}$ resp. $\mathbf{y}^{(k)}$ under the projections onto the approximation and detail spaces are vectors of length N
  - The lengths of the coefficient vectors $\mathbf{a}^{(k)}$ and $\mathbf{d}^{(k)}$ are $N/2^k$.
  - The transformation given by

    $$\mathbf{x} \longmapsto \begin{bmatrix} \mathbf{a}^{(k)} & \mathbf{d}^{(k)} & \mathbf{d}^{(k-1)} & \ldots & d^{(1)} \end{bmatrix}^\top$$

    is the k-level wavelet transform
  - The representation

    $$\mathbf{x} = \mathbf{x}^{(k)} + \mathbf{y}^{(k)} + \mathbf{y}^{(k-1)} + \cdots + y^{(1)}$$

    is the decomposition of the signal $\mathbf{x}$ using projections onto the approximation and wavelet spaces