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The CDF-7/9 filter pair

Construction

Starting with the Daubechies polynomial

```
Daub[N_, z_] := Sum[Binomial[N + k, k] z^k, {k, 0, N}]
```

```
Daub[3, z]
```

$$1 + 4\,z + 10\,z^2 + 20\,z^3$$

Determining the complex roots of this polynomial

```
roots = NSolve[Daub[3, z] == 0, z]
```

$$\{\{z \to -0.342384\}, \{z \to -0.078808 - 0.373931\,i\}, \{z \to -0.078808 + 0.373931\,i\}\}$$

There is one real root and two complex conjugate roots. So the Daubechies polynomial can be factored into two polynomials with real coefficients: a linear polynomial *p(z)* and a quadratic polynomial *q(z)*. The parameter *a* (which is the leading coefficient of the polynomial *p(z)*) will be fixed later.

```
p[z_] = a (z - (z /. roots[[1]]))
```

$$a\,(0.342384 + z)$$

```
q[z_] = (20 / a) (z - (z /. roots[[2]])) (z - (z /. roots[[3]]))
```

$$\frac{1}{a}\,20\,((0.078808 - 0.373931\,i) + z)\,((0.078808 + 0.373931\,i) + z)$$

```
Expand[%]
```

$$\frac{2.9207 + 0.\,i}{a} + \frac{(3.15232 + 0.\,i)\,z}{a} + \frac{20\,z^2}{a}$$

```
Chop[%]
```

$$\frac{2.9207}{a} + \frac{3.15232\,z}{a} + \frac{20\,z^2}{a}$$

The frequency representation of the H-filter that goes with the factor *p(z)*

```
H[w_] = Sqrt[2] Cos[w / 2]^4 p[Sin[w / 2]^2]
```

$$\sqrt{2}\;a\,\text{Cos}\left[\frac{w}{2}\right]^4\left(0.342384 + \text{Sin}\left[\frac{w}{2}\right]^2\right)$$

The frequency representation of the K-filter that goes with the factor *q(z)*

```
K[w_] = Chop[Expand[Sqrt[2] Cos[w / 2]^4 q[Sin[w / 2]^2]]]
```

$$\frac{4.13049\,\text{Cos}\left[\frac{w}{2}\right]^4}{a} + \frac{1}{a}\,4.45805\,\text{Cos}\left[\frac{w}{2}\right]^4\text{Sin}\left[\frac{w}{2}\right]^2 + \frac{1}{a}\,20\,\sqrt{2}\;\text{Cos}\left[\frac{w}{2}\right]^4\text{Sin}\left[\frac{w}{2}\right]^4$$

Evaluating *H(0)* (which should be $\sqrt{2}$)

```
H[0]
```

$$0.484204\,a$$

```
const = Solve[{H[0] == Sqrt[2]}, {a}]
```

$$\{\{a \to 2.9207\}\}$$

The Fourier series if the H-filter after fixing the constant

```
H[w_] = H[w] /. const[[1]]
```

$$4.13049 \cos\left[\frac{w}{2}\right]^4 \left(0.342384 + \sin\left[\frac{w}{2}\right]^2\right)$$

The Fourier series of the K-filter after fixing the constant

```
K[w_] = K[w] /. const[[1]]
```

$$1.41421 \cos\left[\frac{w}{2}\right]^4 + 1.52637 \cos\left[\frac{w}{2}\right]^4 \sin\left[\frac{w}{2}\right]^2 + 9.68408 \cos\left[\frac{w}{2}\right]^4 \sin\left[\frac{w}{2}\right]^4$$

Checking that all is correct

```
K[0]
```

$$1.41421$$

Checking orthogonality conditions for the CDF-7/9 filter pair

```
H[ω] Conjugate[K[ω]] + H[ω + Pi] Conjugate[K[ω + Pi]]
```

$$4.13049 \cos\left[\frac{\omega}{2}\right]^4 \left(\text{Conjugate}\left[1.52637 \cos\left[\frac{\omega}{2}\right]^4 \sin\left[\frac{\omega}{2}\right]^2 + 9.68408 \cos\left[\frac{\omega}{2}\right]^4 \sin\left[\frac{\omega}{2}\right]^4\right] + 1.41421 \cos\left[\frac{\text{Conjugate}[\omega]}{2}\right]^4\right) \left(0.342384 + \sin\left[\frac{\omega}{2}\right]^2\right) + 4.13049 \cos\left[\frac{\pi + \omega}{2}\right]^4 \left(\text{Conjugate}\left[1.52637 \cos\left[\frac{\pi + \omega}{2}\right]^4 \sin\left[\frac{\pi + \omega}{2}\right]^2 + 9.68408 \cos\left[\frac{\pi + \omega}{2}\right]^4 \sin\left[\frac{\pi + \omega}{2}\right]^4\right] + 1.41421 \cos\left[\frac{1}{2}\left(\pi + \text{Conjugate}[\omega]\right)\right]^4\right) \left(0.342384 + \sin\left[\frac{\pi + \omega}{2}\right]^2\right)$$

```
Assuming[ω ∈ Reals, Simplify[%]]
```

$$4.13049 \cos\left[\frac{\omega}{2}\right]^4 \left(0.342384 + \sin\left[\frac{\omega}{2}\right]^2\right) \left(1.41421 \cos\left[\frac{\omega}{2}\right]^4 + \left(0.605255 + 0.0953979 \csc\left[\frac{\omega}{2}\right]^2\right) \sin[\omega]^4\right) + 4.13049 \left(0.342384 + \cos\left[\frac{\omega}{2}\right]^2\right) \sin\left[\frac{\omega}{2}\right]^4 \left(1.41421 \sin\left[\frac{\omega}{2}\right]^4 + \left(0.605255 + 0.0953979 \sec\left[\frac{\omega}{2}\right]^2\right) \sin[\omega]^4\right)$$

```
FullSimplify[Expand[%]]
```

$$2. + 3.33067 \times 10^{-16} \cos[2\omega] + 2.77556 \times 10^{-17} \cos[3\omega] - 2.08167 \times 10^{-17} \cos[4\omega] - 3.46945 \times 10^{-18} \cos[5\omega]$$

```
Chop[%]
```

$$2.$$

Properties of the CDF-7/9 filter pair

```
TrigToExp[H[ω]]
```

$$0.788486 + 0.418092\, e^{-i\omega} + 0.418092\, e^{i\omega} - 0.0406894\, e^{-2i\omega} - 0.0406894\, e^{2i\omega} - 0.0645389\, e^{-3i\omega} - 0.0645389\, e^{3i\omega}$$

```
hcoeffs = CoefficientList[Expand[Exp[3 I ω] %], E^{I ω}]
```

```
{-0.0645389, -0.0406894, 0.418092, 0.788486, 0.418092, -0.0406894, -0.0645389}
```

The filter coefficients of the H-filter of length 7

```
hfilter = Table[{k - 4, hcoeffs[[k]]}, {k, 1, 7}];
TableForm[hfilter, TableAlignments -> "."]
```

```
-3    -0.0645389
-2    -0.0406894
-1     0.418092
 0     0.788486
 1     0.418092
 2    -0.0406894
 3    -0.0645389
```

```
TrigToExp[K[ω]]
```

$$0.852699 + 0.377403\, e^{-i\omega} + 0.377403\, e^{i\omega} - 0.110624\, e^{-2i\omega} - 0.110624\, e^{2i\omega} - 0.0238495\, e^{-3i\omega} - 0.0238495\, e^{3i\omega} + 0.0378285\, e^{-4i\omega} + 0.0378285\, e^{4i\omega}$$

```
kcoeffs = Re[CoefficientList[Expand[Exp[4 I ω] %], E^{I ω}]]
```

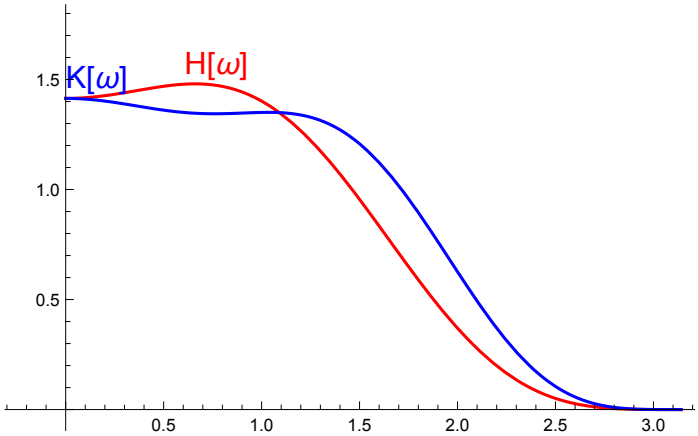
```
{0.0378285, -0.0238495, -0.110624, 0.377403,
 0.852699, 0.377403, -0.110624, -0.0238495, 0.0378285}
```

The filter coefficients of the K-filter of length 9

```
kfilter = Table[{k - 5, kcoeffs[[k]]}, {k, 1, 9}];
TableForm[kfilter, TableAlignments -> "."]
```

```
-4     0.0378285
-3    -0.0238495
-2    -0.110624
-1     0.377403
 0     0.852699
 1     0.377403
 2    -0.110624
 3    -0.0238495
 4     0.0378285
```

```
Plot[{H[ω], K[ω]}, {ω, 0, Pi}, PlotStyle → {Red, Blue}, PlotLabels →
  Placed[{Style["      H[ω]", Red, 16], Style["K[ω]", Blue, 16]}, {Above, Above}]]
```



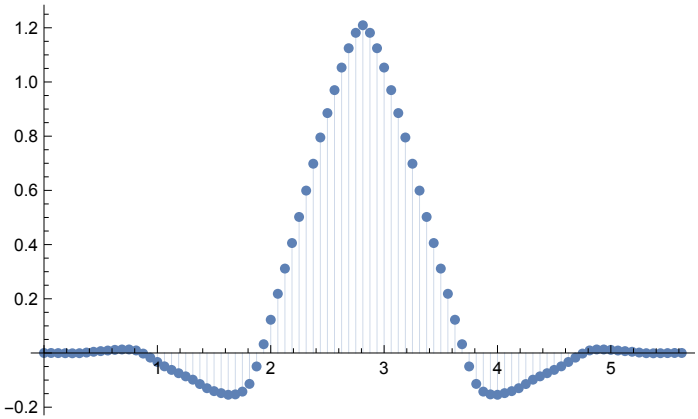
Procedures for the cascade algorithm

```
cascadephidis[h_, n_] :=
 Module[{degrees, min, max, sum, hmodif, hpol, hpols, tbl, clist, z},
  (*Approximation of the scaling function belonging
   to the filter h by using the cascade algorithm*)
  degrees = Map[First[#] &, h];
  min = Min[degrees];
  max = Max[degrees];
  sum = Last[Total[h]];
  hmodif = Map[{First[#], Last[#] / sum} &, h];
  hpol := Apply[Plus, Map[Last[#] z^First[#] &, hmodif]];
  hpols := Expand[Product[hpol /. z → z^(2^t), {t, 0, n - 1}]];
  clist = 2^n CoefficientList[z^(-min (2^n - 1)) hpols, z];
  tbl = Table[{(k - 1) 2^(-n) , clist[[k]]}, {k, 1, Length[clist]}];
  ListPlot[tbl, PlotRange → All, Filling → Axis, PlotStyle → PointSize[0.015]]
 ]

cascadepsidis[h_, n_] :=
 Module[{degrees, min, max, sum, hmodif, gpol, hpol, hpols, clist, tbl, z},
  (*Approximation of the wavelet function belonging
   to the filter h by using the cascade algorithm*)
  degrees = Map[First[#] &, h];
  min = Min[degrees];
  max = Max[degrees];
  sum = Last[Total[h]];
  hmodif = Map[{First[#], Last[#] / sum} &, h];
  hpol := Apply[Plus, Map[Last[#] z^First[#] &, hmodif]];
  gpol := hpol /. (z → -1 / z);
  hpols :=
   Expand[Product[hpol /. z → z^(2^t), {t, 0, n - 2}] (gpol /. z → z^(2^(n - 1)))];
  clist = 2^n CoefficientList[z^(-min (2^(n - 1) - 1) + 2^(n - 1) max) hpols, z];
  tbl = Table[{ (k - 1) 2^(-n) , clist[[k]]}, {k, 1, Length[clist]}];
  ListPlot[tbl, PlotRange → All, Filling → Axis, PlotStyle → PointSize[0.015]]
 ]
```

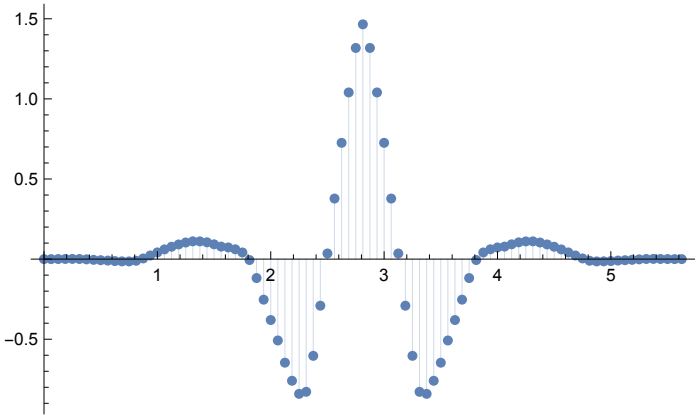
The scaling function belonging to the H-filter of length 7

```
cascadephidis[hfilter, 4]
```



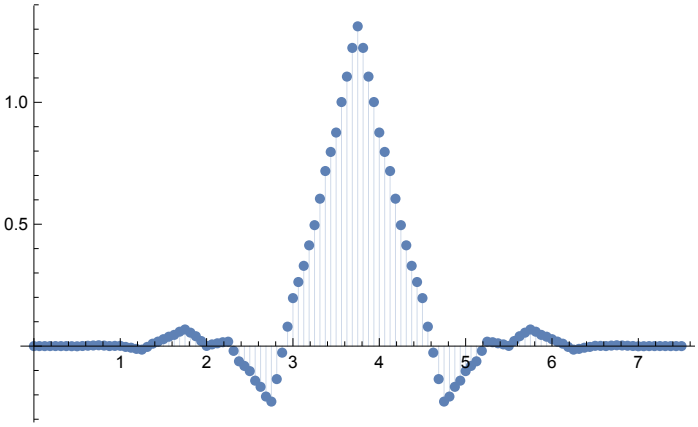
The wavelet function belonging to the H-filter of length 7

```
cascadepsidis[hfilter, 4]
```



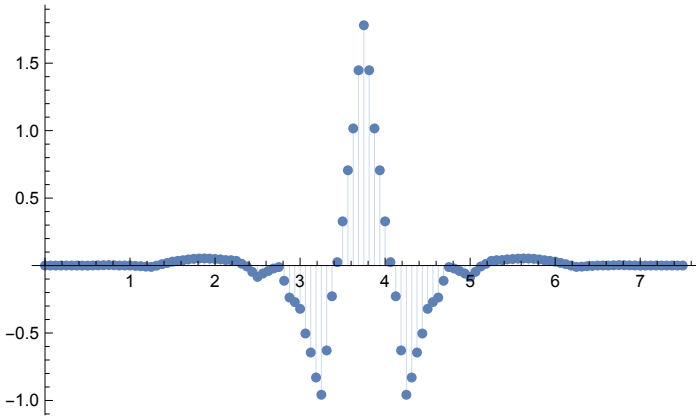
The scaling function belonging to the K-filter of length 9

```
cascadephidis[kfilter, 4]
```



The wavelet function belonging to the K-filter of length 9

```
cascadepsidis[kfilter, 4]
```



Low-pass conditions for the H-filter

```
{H'[ω], H'[ω] /. ω → Pi}
```

$$\left\{4.13049\ \text{Cos}\left[\frac{\omega}{2}\right]^5 \text{Sin}\left[\frac{\omega}{2}\right] - 8.26098\ \text{Cos}\left[\frac{\omega}{2}\right]^3 \text{Sin}\left[\frac{\omega}{2}\right] \left(0.342384 + \text{Sin}\left[\frac{\omega}{2}\right]^2\right),\ 0.\right\}$$

```
{H''[ω], H''[ω] /. ω → Pi}
```

$$\left\{2.06524\ \text{Cos}\left[\frac{\omega}{2}\right]^6 - 18.5872\ \text{Cos}\left[\frac{\omega}{2}\right]^4 \text{Sin}\left[\frac{\omega}{2}\right]^2 - 4.13049\ \text{Cos}\left[\frac{\omega}{2}\right]^4 \left(0.342384 + \text{Sin}\left[\frac{\omega}{2}\right]^2\right) + 12.3915\ \text{Cos}\left[\frac{\omega}{2}\right]^2 \text{Sin}\left[\frac{\omega}{2}\right]^2 \left(0.342384 + \text{Sin}\left[\frac{\omega}{2}\right]^2\right),\ 0.\right\}$$

```
{H'''[ω], H'''[ω] /. ω → Pi}
```

$$\left\{-28.9134\ \text{Cos}\left[\frac{\omega}{2}\right]^5 \text{Sin}\left[\frac{\omega}{2}\right] + 49.5659\ \text{Cos}\left[\frac{\omega}{2}\right]^3 \text{Sin}\left[\frac{\omega}{2}\right]^3 + 20.6524\ \text{Cos}\left[\frac{\omega}{2}\right]^3 \text{Sin}\left[\frac{\omega}{2}\right] \left(0.342384 + \text{Sin}\left[\frac{\omega}{2}\right]^2\right) - 12.3915\ \text{Cos}\left[\frac{\omega}{2}\right] \text{Sin}\left[\frac{\omega}{2}\right]^3 \left(0.342384 + \text{Sin}\left[\frac{\omega}{2}\right]^2\right),\ 0.\right\}$$

```
{H''''[ω], H''''[ω] /. ω → Pi}
```

$$\left\{-14.4567\ \text{Cos}\left[\frac{\omega}{2}\right]^6 + 167.285\ \text{Cos}\left[\frac{\omega}{2}\right]^4 \text{Sin}\left[\frac{\omega}{2}\right]^2 - 86.7403\ \text{Cos}\left[\frac{\omega}{2}\right]^2 \text{Sin}\left[\frac{\omega}{2}\right]^4 + 10.3262\ \text{Cos}\left[\frac{\omega}{2}\right]^4 \left(0.342384 + \text{Sin}\left[\frac{\omega}{2}\right]^2\right) - 49.5659\ \text{Cos}\left[\frac{\omega}{2}\right]^2 \text{Sin}\left[\frac{\omega}{2}\right]^2 \left(0.342384 + \text{Sin}\left[\frac{\omega}{2}\right]^2\right) + 6.19573\ \text{Sin}\left[\frac{\omega}{2}\right]^4 \left(0.342384 + \text{Sin}\left[\frac{\omega}{2}\right]^2\right),\ 8.31705\right\}$$

So the H-filter of length 7 has 4 vanishing moments

Low-pass conditions for the K-filter

```
{K'[ω], K'[ω] /. ω → Pi}
```

$$\left\{-2.82843\,\text{Cos}\left[\frac{\omega}{2}\right]^3\text{Sin}\left[\frac{\omega}{2}\right]+1.52637\,\text{Cos}\left[\frac{\omega}{2}\right]^5\text{Sin}\left[\frac{\omega}{2}\right]-3.05273\,\text{Cos}\left[\frac{\omega}{2}\right]^3\text{Sin}\left[\frac{\omega}{2}\right]^3+\right.$$
$$\left.19.3682\,\text{Cos}\left[\frac{\omega}{2}\right]^5\text{Sin}\left[\frac{\omega}{2}\right]^3-19.3682\,\text{Cos}\left[\frac{\omega}{2}\right]^3\text{Sin}\left[\frac{\omega}{2}\right]^5,\,0.\right\}$$

```
{K''[ω], K''[ω] /. ω → Pi}
```

$$\left\{-1.41421\,\text{Cos}\left[\frac{\omega}{2}\right]^4+0.763183\,\text{Cos}\left[\frac{\omega}{2}\right]^6+4.24264\,\text{Cos}\left[\frac{\omega}{2}\right]^2\text{Sin}\left[\frac{\omega}{2}\right]^2-\right.$$
$$8.39501\,\text{Cos}\left[\frac{\omega}{2}\right]^4\text{Sin}\left[\frac{\omega}{2}\right]^2+29.0523\,\text{Cos}\left[\frac{\omega}{2}\right]^6\text{Sin}\left[\frac{\omega}{2}\right]^2+4.5791\,\text{Cos}\left[\frac{\omega}{2}\right]^2\text{Sin}\left[\frac{\omega}{2}\right]^4-$$
$$\left.96.8408\,\text{Cos}\left[\frac{\omega}{2}\right]^4\text{Sin}\left[\frac{\omega}{2}\right]^4+29.0523\,\text{Cos}\left[\frac{\omega}{2}\right]^2\text{Sin}\left[\frac{\omega}{2}\right]^6,\,0.\right\}$$

```
{K'''[ω], K'''[ω] /. ω → Pi}
```

$$\left\{7.07107\,\text{Cos}\left[\frac{\omega}{2}\right]^3\text{Sin}\left[\frac{\omega}{2}\right]-10.6846\,\text{Cos}\left[\frac{\omega}{2}\right]^5\text{Sin}\left[\frac{\omega}{2}\right]+29.0523\,\text{Cos}\left[\frac{\omega}{2}\right]^7\text{Sin}\left[\frac{\omega}{2}\right]-\right.$$
$$4.24264\,\text{Cos}\left[\frac{\omega}{2}\right]\text{Sin}\left[\frac{\omega}{2}\right]^3+25.9482\,\text{Cos}\left[\frac{\omega}{2}\right]^3\text{Sin}\left[\frac{\omega}{2}\right]^3-280.838\,\text{Cos}\left[\frac{\omega}{2}\right]^5\text{Sin}\left[\frac{\omega}{2}\right]^3-$$
$$\left.4.5791\,\text{Cos}\left[\frac{\omega}{2}\right]\text{Sin}\left[\frac{\omega}{2}\right]^5+280.838\,\text{Cos}\left[\frac{\omega}{2}\right]^3\text{Sin}\left[\frac{\omega}{2}\right]^5-29.0523\,\text{Cos}\left[\frac{\omega}{2}\right]\text{Sin}\left[\frac{\omega}{2}\right]^7,\,0.\right\}$$

```
{K''''[ω], K''''[ω] /. ω → Pi}
```

$$\left\{3.53553\,\text{Cos}\left[\frac{\omega}{2}\right]^4-5.34228\,\text{Cos}\left[\frac{\omega}{2}\right]^6+14.5261\,\text{Cos}\left[\frac{\omega}{2}\right]^8-\right.$$
$$16.9706\,\text{Cos}\left[\frac{\omega}{2}\right]^2\text{Sin}\left[\frac{\omega}{2}\right]^2+65.6337\,\text{Cos}\left[\frac{\omega}{2}\right]^4\text{Sin}\left[\frac{\omega}{2}\right]^2-522.941\,\text{Cos}\left[\frac{\omega}{2}\right]^6\text{Sin}\left[\frac{\omega}{2}\right]^2+$$
$$2.12132\,\text{Sin}\left[\frac{\omega}{2}\right]^4-50.3701\,\text{Cos}\left[\frac{\omega}{2}\right]^2\text{Sin}\left[\frac{\omega}{2}\right]^4+1404.19\,\text{Cos}\left[\frac{\omega}{2}\right]^4\text{Sin}\left[\frac{\omega}{2}\right]^4+$$
$$\left.2.28955\,\text{Sin}\left[\frac{\omega}{2}\right]^6-522.941\,\text{Cos}\left[\frac{\omega}{2}\right]^2\text{Sin}\left[\frac{\omega}{2}\right]^6+14.5261\,\text{Sin}\left[\frac{\omega}{2}\right]^8,\,18.937\right\}$$

So the K-filter of length 9 has 4 vanishing moments