Multiresolution Analysis (MRA)

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Multiresolutions analysis (MRA)

- was invented in 1988 by Stephane MALLAT in his Ph.D. thesis *Multiresolution and Wavelets* (University of Pennsylvania)
- is an elegant theoretical framework for the study of wavelets and wavelet transforms
- is considered to be ***the*** central concept which integrates the many facets of wavelet transforms

- Definition
  An MRA (*multiresolution analysis*) consists of a family $\{V_j\}_{j\in\mathbb{Z}}$ of subspaces of $\mathcal{L}^2(\mathbb{R})$ satisfying the following properties:
  1. "nesting": $V_j \subseteq V_{j+1}$ $(j \in \mathbb{Z})$
  2. "density" : $\overline{span}\{V_j\}_{j\in\mathbb{Z}} = \mathcal{L}^2(\mathbb{R})$
  3. "separation": $\bigcap\{V_j\}_{j\in\mathbb{Z}} = \{0\}$
  4. "scaling":
     $f(t) \in V_0 \;\Leftrightarrow\; (D_{2^j} f)(t) = 2^{j/2} f(2^j t) \in V_j \qquad (f \in \mathcal{L}^2(\mathbb{R}), j \in \mathbb{Z})$
  5. "scaling function":
     There exists a function $\phi \in V_0$ s.th. the family of its integer translates

     $$\{\, T_k\phi(t)\,\}_{k\in\mathbb{Z}} = \{\,\phi(t-k)\,\}_{k\in\mathbb{Z}}$$

     forms a complete ON-basis of $V_0 = \overline{span}\{\,T_k\phi\,\}_{k\in\mathbb{Z}}$ (ONST)

- ONST-Example:
  - Consider the function $\phi(t) = \mathrm{sinc}(t) = \dfrac{\sin(\pi t)}{\pi t}$
  - Are the integer translates $(T_k\phi)(t) = \phi(t-k)\ \ (k \in \mathbb{Z})$ orthogonal to each other?
  - The answer is not obvious from looking at the graphs!
  - How to prove orthogonality?
  - Recipe: Go to the frequency domain! (using PP)
  - Recall: $\widehat{\phi}(s) = b(s) = \mathbf{1}_{[-1/2,1/2)}(s)$ (the box function)

$$\langle\, \phi \,|\, T_k\phi \,\rangle = \langle\, \widehat{\phi} \,|\, \widehat{T_k\phi} \,\rangle = \langle\, b(s) \,|\, e^{-2\pi iks} b(s) \,\rangle$$

$$= \int_{-1/2}^{1/2} e^{-2\pi iks}\, ds = \delta_{0,k}$$

- Reminder:

$$\phi(t) \text{ satisfies (ONST)} \iff \sum_{n\in\mathbb{Z}} |\widehat{\phi}(s+n)|^2 =\equiv 1$$

$$\text{Proof :} \quad \langle\, f \mid T_k f \,\rangle = \langle\, \widehat{f} \mid \widehat{T_k f} \,\rangle = \int_{\mathbb{R}} \widehat{f}(s)\, \overline{\widehat{f}(s)}\, e^{2\pi iks} ds$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} \left|\widehat{f(s)}\right|^2 e^{2\pi iks} ds$$

$$= \sum_{n\in\mathbb{Z}} \int_0^1 \left|\widehat{f}(s+n)\right|^2 e^{2\pi iks} ds$$

$$= \int_0^1 \sum_{n\in\mathbb{Z}} \left|\widehat{f}(s+n)\right|^2 e^{2\pi iks} ds$$

Hence in terms of Fourier series

$$\sum_{k\in\mathbb{Z}} \langle\, f \mid T_k f \,\rangle\, e^{-2\pi iks} = \sum_{n\in\mathbb{Z}} \left|\widehat{f}(s+n)\right|^2$$

- Consequences

1. The vector spaces $(V_j)_{j\in\mathbb{Z}}$ are ordered by inclusion

$$\{0\} \swarrow \cdots \subseteq V_{-2} \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \nearrow \mathcal{L}^2(\mathbb{R})$$

2. For each $j \in \mathbb{Z}$ family of dilated and translated functions $\{\, \phi_{j,k}(t) \,\}_{k\in\mathbb{Z}}$, defined by

$$\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k) = (D_{2^j}\, T_k\, \phi)(t),$$

forms a complete ON-Basis (Hilbert basis) of the approximation space

$$V_j = \overline{span}\{\, \phi_{j,k} \,\}_{k\in\mathbb{Z}} \quad (j \in \mathbb{Z})$$

3. From $V_0 \subseteq V_1$ it follows that there exists a (unique) ℓ^2-sequence $\boldsymbol{h} = (h_k)_{k\in\mathbb{Z}}$ of complex numbers s.th.

$$(S) \qquad \phi(t) = \sum_{k\in\mathbb{Z}} h_k\, \phi_{1,k}(t)$$

This identity is the *scaling identity* of the MRA, the sequence $\boldsymbol{h} = (h_k)_{k\in\mathbb{Z}}$ is the *scaling filter* of the MRA

- Remarks
  Properties involving V_0 and $\phi(t)$ carry over to all scaling levels by using dilation, e.g.,

$$V_0 \ni f(t) = \sum_{k \in \mathbb{Z}} f_k \cdot (T_k \phi)(t) \iff V_j \ni (D_{2^j} f)(t) = \sum_{k \in \mathbb{Z}} f_k \cdot (D_{2^j} T_k \phi)(t)$$

  so each V_j is a dilated copy of V_0,
- and thus orthonormality is preserved

$$\begin{aligned} \langle \phi_{j,k} \,|\, \phi_{j,\ell} \rangle &= 2^j \int_{\mathbb{R}} \phi(2^j t - k) \overline{\phi(2^j t - \ell)} \, dt \\ &= \int_{\mathbb{R}} \phi(t - k) \overline{\phi(t - \ell)} \, dt = \langle \phi_{0,k} \,|\, \phi_{0,\ell} \rangle = \delta_{k,\ell} \end{aligned}$$

- From the scaling identity (S) and orthogonality one gets immediately

$$h_k = \langle \phi \,|\, \phi_{1,k} \rangle = \sqrt{2} \int_{\mathbb{R}} \phi(t) \, \overline{\phi(2t-k)} \, dt$$

- and for all $j, \ell \in \mathbb{Z}$

$$\begin{aligned} \phi_{j,\ell}(t) &= 2^{j/2} \phi(2^j t - \ell) \\ &= 2^{j/2} \sum_{k \in \mathbb{Z}} h_k \, \phi_{1,k}(2^j t - \ell) \\ &= 2^{(j+1)/2} \sum_{k \in \mathbb{Z}} h_k \, \phi(2^{j+1} - 2\ell - k) \\ &= \sum_{k \in \mathbb{Z}} h_k \, \phi_{j+1,2\ell+k}(t) = \sum_{k \in \mathbb{Z}} h_{k-2\ell} \, \phi_{j+1,k}(t) \end{aligned}$$

- so that the *scaling coefficients* $a_{j,\ell} = \langle f \,|\, \phi_{j,\ell} \rangle$ of $f \in \mathcal{L}^2$ satisfy

$$a_{j,\ell} = \langle f \,|\, \phi_{j,\ell} \rangle = \sum_{k \in \mathbb{Z}} h_{k-2\ell} \langle f \,|\, \phi_{j+1,k}(t) \rangle = \sum_{k \in \mathbb{Z}} h_{k-2\ell} \, a_{j+1,k}$$

- The *wavelet function* $\psi(t)$ of a MRA is defined in terms of the scaling function $\phi(t)$ as

$$(W) \qquad \psi(t) = \sum_{k\in\mathbb{Z}} g_k\,\phi_{1,k}(t) \;\text{ where }\; g_k = (-1)^k\,\overline{h_{1-k}}$$

- The sequence $\boldsymbol{g} = (g_k)_{k\in\mathbb{Z}}$ is the *wavelet filter* belonging to the MRA
- The wavelet functions $\psi_{j,\ell}$ ($j, \ell \in \mathbb{Z}$ are defined as usual
- The *wavelet coefficients* $d_{j,\ell} = \langle\, f \,|\, \psi_{j,\ell} \,\rangle$ of $f \in \mathcal{L}^2$ satisfy

$$d_{j,\ell} = \langle\, f \,|\, \psi_{j,\ell} \,\rangle = \sum_{k\in\mathbb{Z}} g_{k-2\ell}\,\langle\, f \,|\, \phi_{j+1,k}(t) \,\rangle = \sum_{k\in\mathbb{Z}} g_{k-2\ell}\, a_{j+1,k}$$

- The Discrete Wavelet Transform (DWT) based on the functions $\phi(t)$ and $\psi(t)$ uses these scaling and wavelet identities

$$a_{j,\ell} = \sum_{k\in\mathbb{Z}} h_{k-2\ell}\, a_{j+1,k} \qquad\qquad d_{j,\ell} = \sum_{k\in\mathbb{Z}} g_{k-2\ell}\, a_{j+1,k}$$

- Theorem
  1. For each $j \in \mathbb{Z}$ the family of wavelet functions $\{ \psi_{j,k} \}_{k \in \mathbb{Z}}$ with
  $$\psi_{j,k}(t) = 2^{j/2}\, \psi(2^j t - k) = (D_{2^j} T_k \psi)(t)$$
  is a complete ON-Basis (Hilbert basis) of the wavelet (detail) space
  $$W_j = \overline{span}\{ \psi_{j,k} \}_{k \in \mathbb{Z}}$$
  2. For all $j \in \mathbb{Z}$ the space W_j is the orthogonal complement of V_j in V_{j+1}:
  $$V_{j+1} = W_j \oplus V_j \qquad W_j \perp V_j$$
  3. For every $J \in \mathbb{Z}$ one has the direct product decomposition
  $$\mathcal{L}^2(\mathbb{R}) = V_J \oplus \bigoplus_{j \geq J} W_j$$
  4. The family $\{ \psi_{j,k} \}_{j,k \in \mathbb{Z}}$ is a complete ON-basis (Hilbert basis) of $\mathcal{L}^2(\mathbb{R})$
  $$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j$$

- Remarks
  1. Functions in V_j and W_j have resolution level $\geq 2^{-j}$
  2. Orthogonal projections on approximation and detail subspaces

$$\text{approximation} \quad P_j : \mathcal{L}^2(\mathbb{R}) \to V_j : f \mapsto \sum_{k \in \mathbb{Z}} \langle\, f \mid \phi_{j,k} \,\rangle\, \phi_{j,k}$$

$$\text{detail} \quad Q_j : \mathcal{L}^2(\mathbb{R}) \to W_j : f \mapsto \sum_{k \in \mathbb{Z}} \langle\, f \mid \psi_{j,k} \,\rangle\, \psi_{j,k}$$

     where $Q_j = P_{j+1} - P_j$
  3. For all $j > m$ one has the wavelet decomposition

$$V_{j+1} = V_m \oplus W_m \oplus W_{m+1} \oplus \cdots \oplus W_j$$

  4. The “density” and “separation” requirements for an MRA translate into

$$\lim_{j \to \infty} P_j\, f = f \quad \text{und} \quad \lim_{j \to -\infty} P_j\, f = 0$$

     w.r.t. $\mathcal{L}^2$-convergence

Example (1): The HAAR-MRA

- The scaling function is

$$\phi(t) = \mathbf{1}_{[0,1)}(t)$$

- For $j \in \mathbb{Z}$ the approximation space

$$V_j = \overline{span}\{\, \phi_{j,k}(t) \,\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}^2(\mathbb{R})$$

  consists of the $\mathcal{L}^2$-step functions with step width 2^{-j}
- $\{\, \phi_{j,k}(t) \,\}_{k\in\mathbb{Z}}$ is obviously an ON-Basis of V_j
- Density (fact about approximation by step functions):

$$\lim_{j\to\infty} V_j = \mathcal{L}^2(\mathbb{R})$$

- Separation: an $\mathcal{L}^2$-function $f \in \bigcap_{j\in\mathbb{Z}} V_j$ which is constant on arbitrarily long intervals must vanish identically on $\mathbb{R}$

- Scaling filter coefficients

$$h_0 = \frac{1}{\sqrt{2}},\ h_1 = \frac{1}{\sqrt{2}},\ h_k = 0\ (k \neq 0, 1)$$

- Scaling identity

$$\phi(t) = \frac{1}{\sqrt{2}}\left(\phi_{0,0}(t) + \phi_{0,1}(t)\right) = \phi(2t) + \phi(2t-1)$$

- Wavelet filter coefficients

$$g_0 = \frac{1}{\sqrt{2}},\ g_1 = -\frac{1}{\sqrt{2}},\ g_k = 0\ (k \neq 0, 1)$$

- Wavelet identity

$$\begin{aligned}\psi(t) &= \frac{1}{\sqrt{2}}\left(\phi_{0,0}(t) + \phi_{0,1}(t)\right) = \phi(2t) - \phi(2t-1) \\ &= \mathbf{1}_{[0,1/2)}(t) - \mathbf{1}_{[1/2,1)}(t)\end{aligned}$$

- Fourier transforms

$$\widehat{\phi}(s) = e^{-i\pi s}\mathrm{sinc}(s) \qquad \widehat{\psi}(s) = i \cdot e^{-i\pi s}\sin(\pi s/2)\,\mathrm{sinc}(s/2)$$

- Examples (2)

  The *Daubechies*, *Coiflet*, and many other orthogonal filters of similar type define MRAs with filters of finite length and scaling/wavelet functions with compact support

  - The filters are (of course!) those constructed from orthogonality and low/highpass conditions
  - The scaling functions $\phi(t)$ and the wavelet functions $\psi(t)$ are those functions determined by the cascade algorithm
  - The ONST-property follows because the cascade algorithm preserves orthogonality
  - *Density* and *Separation* do not come automatically, but have to be verified separately

Example (3): The SHANNON-MRA

- SHANNON's sampling theorem motivates to consider

$$V_0 = \{ f \in \mathcal{L}^2(\mathbb{R}) \,;\, \widehat{f}(s) = 0 \text{ for } |s| > 1/2 \}$$

  the space of 1-band-limited functions, and

$$V_j = \{ f \in \mathcal{L}^2(\mathbb{R}) \,;\, \widehat{f}(s) = 0 \text{ for } |s| > 2^{j-1} \}$$

  the space of 2^j-band-limited functions

- The scaling function is

$$\phi(t) = \mathrm{sinc}(t) = \frac{\sin(\pi\, t)}{\pi\, t}$$

- The FT of the scaling function is the box function

$$\widehat{\phi}(s) = \mathbf{1}_{[-1/2,1/2)}(s)$$

- The family $\{ T_k\phi(t) \}_{k\in\mathbb{Z}} \subseteq V_0$ is an ONST in V_0 (remember the previous example)
- SHANNON's sampling theorem says precisely this:

$$V_0 = \overline{span}\{ T_k\phi(t) \}_{k\in\mathbb{Z}}$$

- The Shannon wavelet function is

$$\psi(t) = \frac{\sin(2\pi t) - \cos(\pi t)}{\pi(t - 1/2)} = \frac{\sin(\pi(t - 1/2))}{\pi(t - 1/2)} (1 - 2\sin(\pi t))$$

- with its FT

$$\widehat{\psi}(t) = -e^{-i\pi s} \left(\mathbf{1}_{[-1,-1/2)}(s) + \mathbf{1}_{[1/2,1)}(s)\right)$$

- Note:
  - $\phi(t)$ and $\psi(t)$ are infinitely differentiable functions with infinite support
  - $\widehat{\phi}(t)$ and $\widehat{\psi}(t)$ discontinuous functions with compact support
  - The scaling and wavelet filters have infinite length (with quite simple coefficients)
- The situation is precisely the converse to that of the HAAR-MRA

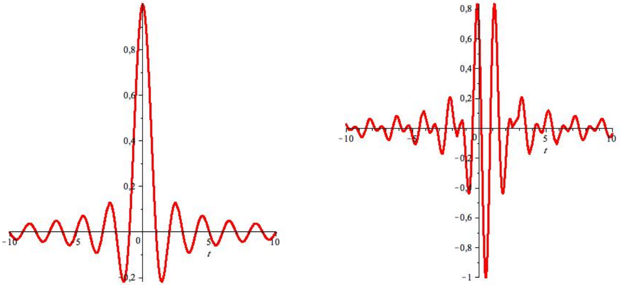


Figure: Shannon Scaling function and Shannon wavelet function

Example (4): The piecewise-linear MRA

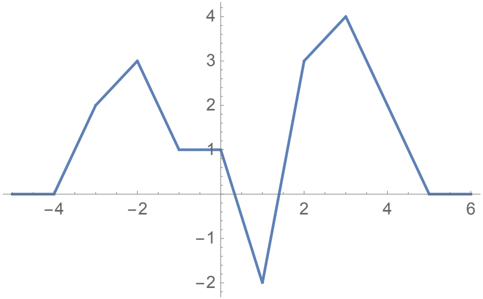
- Continuous alternative to the HAAR-MRA:

  V_0 contains the <u>continuous</u> $\mathcal{L}^2$-functions which are (affine-)linear on any interval $I_{0,k} = [k, k+1)$, $(k \in \mathbb{Z})$,i.e.,

  $$V_0 = \left\{ f \in \mathcal{L}^2(\mathbb{R})\,;\, f \text{ continuous on } \mathbb{R} \text{ and linear on all } I_{0,k}\ (k \in \mathbb{Z}) \right\}$$

- so that for any $j \in \mathbb{Z}$

  $$V_j = \left\{ f \in \mathcal{L}^2(\mathbb{R})\,;\, f \text{ continuous on } \mathbb{R} \text{ and linear on all } I_{j,k}\ (k \in \mathbb{Z}) \right\}$$



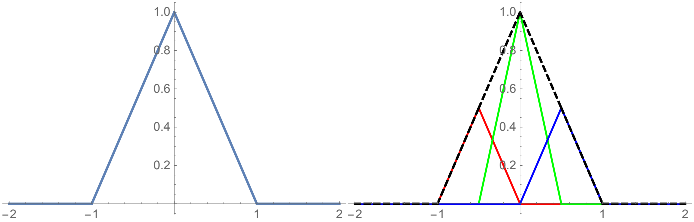
A piecewise-continuous function $f(t)$ defined by the values

k	-4	-3	-3	-1	0	1	2	3	4	5
$f(k)$	0	2	3	1	1	-2	3	4	2	0

- The spaces $(V_j)_{j \in \mathbb{Z}}$ are obviously *nested*
- *Density*: one has to show that any continuous function with compact support can be approximated uniformly as $j \to \infty$ by V_j-functions
- *Separation*: any $\mathcal{L}^2$-function $f \in \bigcap_{j \in \mathbb{Z}} V_j$ must be linear in arbitrarily long intervals.
  This happens only for $f \equiv 0$
- *Scaling* is part of the definition

- What is a *scaling function* $\phi(t) \in V_0$ for this MRA?
  - The "obvious" candidate is the "hat" function

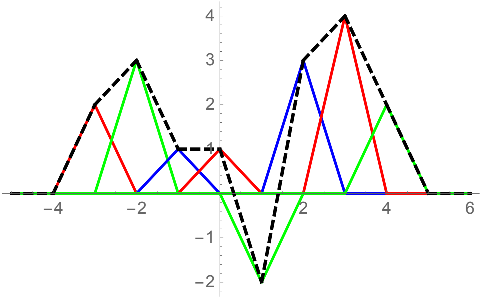
$$\phi(t) = (1 - |t|)\, \mathbf{1}_{[-1,1)}(t)$$

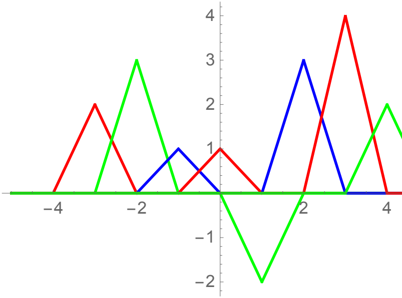


  - It satisfies the scaling equation

$$\phi(t) = \frac{1}{2}\phi(2t-1) + \phi(2t) + \frac{1}{2}\phi(2t+1)$$

- The integer translates $T_k\phi(t)$ $(k \in \mathbb{Z})$ of the hat function can be used to generate V_0





The piecewise-linear function $f(t)$ represented as

$$2\phi(t+3)+3\phi(t+2)+\phi(t+1)+\phi(t)-2\phi(t-1)+3\phi(t-2)+4\phi(t-3)+2\phi(t-4)$$

The example illustrates the simple fact:

- Lemma
  If f is continuous function on $\mathbb{R}$ and linear on all intervalls $I_{0,k}$, then for all $t \in \mathbb{R}$:

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\,(T_k\phi)(t) = \sum_{k\in\mathbb{Z}} f(k)\,\phi(t-k)$$

- This is an assertion about pointwise convergence.
  (This convergence is trivial because for any $t \in \mathbb{R}$ at most two summands are $\neq 0$)

- BUT unfortunately the $T_k\phi(t)$ are <u>not always orthogonal</u> :

$$\langle\, T_k\phi \,|\, T_\ell\phi \,\rangle = \begin{cases} 2/3 & \text{if } k = \ell \\ 1/6 & \text{if } |k-\ell| = 1 \\ 0 & \text{otherwise} \end{cases}$$

- **Q**: Can one find another function $\widetilde{\phi}(t) \in V_0$ such that its integer translates are an ONST and generate V_0 ?
- The procedure outlined below is exemplary and can be used in other situations as well

- (still about the scaling function)
  - Lemma
    If f is continuous on $\mathbb{R}$ and linear on all intervalls $I_{0,k}$, then

    $$f(t) = \sum_{k \in \mathbb{Z}} f(k)\,(T_k \phi)(t)$$

    also holds in the sense of $\mathcal{L}^2$-convergence
  - This follows from

    $$\frac{1}{6}\left(|f(n)|^2 + |f(n+1)|^2\right) \le \int_n^{n+1} |f(t)|^2\,dt \le \frac{1}{2}\left(|f(n)|^2 + |f(n+1)|^2\right)$$

    for any function which is linear in the interval $[n, n+1)$
  - Lemma: $V_0 = \overline{span}\{\,T_k \phi\,\}_{k \in \mathbb{Z}}$

- (still about the scaling function)
  - A suitable scaling function $\widetilde{\phi}(t)$ for the piecewise-linear MRA can be found using Fourier transforms
  - Remember the chacterization of ONST

$$\{ T_k \phi \}_{k \in \mathbb{Z}} \text{ is an ONST} \Longleftrightarrow \sum_{n \in \mathbb{Z}} \left| \widehat{\phi}(s+n) \right|^2 \equiv 1$$

  - The translates of $\phi(t)$ visibly do not form an ONST, and this can be quantified by

$$\sum_{n \in \mathbb{Z}} \left| \widehat{\phi}(s+n) \right|^2 = \frac{1}{6} e^{-2\pi i\, s} + \frac{2}{3} + + \frac{1}{6} e^{2\pi i\, s} = \frac{1 + 2\cos^2(\pi\, s)}{3},$$

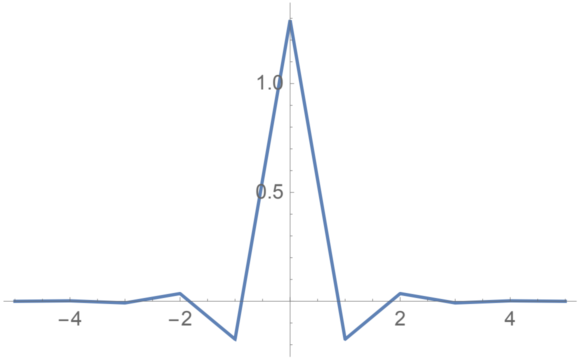
and hence

$$\frac{1}{3} \leq \sum_{n \in \mathbb{Z}} \left| \widehat{\phi}(s+n) \right|^2 \leq 1$$

- (still about the scaling function)
  - If $\widehat{\phi}(s)$ is the FT of $\phi(t)$, define $\widetilde{\phi}(t)$ through its Fourier transform by setting
$$\widehat{\widetilde{\phi}}(s) = \frac{\sqrt{3}}{\sqrt{1 + 2\cos^2 \pi s}}\,\widehat{\phi}(s),$$
  - Then, by construction,
$$\sum_{n\in\mathbb{Z}} \left|\widehat{\widetilde{\phi}}(s+n)\right|^2 \equiv 1$$
    Hence $\{T_k\widetilde{\phi}\}_{k\in\mathbb{Z}}$ is an ONST and is an ON-basis of V_0
    (see a later theorem for justifying this)
  - The modification of the FT given above leads to the desired conclusion

    But unfortunately neither $\widetilde{\phi}(t)$ nor $\widetilde{\psi}(t)$ have a simple analytic form

- The scaling function $\widetilde{\phi}(t)$ for the piecewise-linear MRA



The family of integer translates of $\widetilde{\phi}(t)$ is an ONST for V_0 of this MRA

- General setup:
  - An MRA given by nested approximation spaces $(V_j)_{j\in\mathbb{Z}}$ and a scaling function $\phi(t)$, satisfying the MRA requirements
  - $\boldsymbol{h} = (h_k)_{k\in\mathbb{Z}}$, the scaling filter of the MRA and its Fourier series

$$m_0(s) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} h_k \, e^{-2\pi i\, s}$$

  - $\boldsymbol{g} = (g_k)_{k\in\mathbb{Z}}$, where $g_k = (-1)^k \overline{h_{1-k}}$, the wavelet filter of the MRA and its Fourier series

$$m_1(s) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} g_k \, e^{-2\pi i\, s}$$

  - $\psi(t) = \sum_{k\in\mathbb{Z}} g_k \phi_{1,k}(t)$ the wavelet function of the MRA

- The following assertions are either already known or follow from the definitions and known facts by straighforward (occasionally somewhat tedious) calculations. See the Lecture Notes for details.
- Properties of $\boldsymbol{h} = (h_k)_{k\in\mathbb{Z}}$
    1. $\sum_k h_{k-2\ell}\,\overline{h_k} = \delta_{\ell,0}$ $\qquad |m_0(s)|^2 + |m_0(s+\frac{1}{2})|^2 = 1$
    2. $\sum_k |h_k|^2 = 1$ $\qquad$ case $\ell = 0$ in (1)
    3. $\sum_k h_k = \sqrt{2}$ $\qquad m_0(0) = 1$
    4. $\sum_k h_{2k} = \sum_k h_{2k+1} = 1/\sqrt{2}$ $\qquad m_0(\frac{1}{2}) = 0$

- Properties of the $\boldsymbol{g} = (g_k)_{k\in\mathbb{Z}}$
  5. $\sum_k g_{k-2\ell}\,\overline{g_k} = \delta_{\ell,0}$ $\qquad |m_1(s)|^2 + |m_1(s+\frac{1}{2})|^2 = 1$
  6. $\sum_k |g_k|^2 = 1$ $\qquad$ case $\ell = 0$ in (1)
  7. $\sum_k g_k = 0$ $\qquad m_1(0) = 0$
  8. $\sum_k g_{2k} = -\sum_k g_{2k+1} = 1/\sqrt{2}$ $\qquad m_1(\frac{1}{2}) = 1$
- Properties relating $\boldsymbol{h} = (h_k)_{k\in\mathbb{Z}}$ and $\boldsymbol{g} = (g_k)_{k\in\mathbb{Z}}$
  9. $\sum_k g_{k-2\ell}\,\overline{h_k} = 0$ $\qquad m_0(s)\overline{m_1(s)} + m_0(s+\frac{1}{2})\overline{m_1(s+\frac{1}{2})} = 0$
  10. $\sum_k \left(h_{m-2k}\,\overline{h_{n-2k}} + g_{m-2k}\,\overline{g_{n-2k}}\right) = \delta_{m,n}$

$$m_0(s)\overline{m_0(s+\tfrac{1}{2})} + m_1(s+1)\overline{m_1(s+\tfrac{1}{2})} = 0$$

- Consequences

  1. For each $j \in \mathbb{Z}$ the family $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$ is an orthonormal family of $\mathcal{L}^2$-functions
  2. For each $j \in \mathbb{Z}$ the families $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$ and $\{\phi_{j,k}\}_{k \in \mathbb{Z}}$ are orthogonal to each other, i.e., $W_j \perp V_j$
  3. One has $V_1 = V_0 \oplus W_0$, and generally $V_{j+1} = V_j \oplus W_j$
  4. For $j \neq j'$ one has $W_j \perp W_{j'}$
  5. Thus $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ is an orthonormal family of $\mathcal{L}^2$-functions

- Charcterization of the elements of the subspace V_0
- Theorem

  If $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an ONST and V_0 the $\mathcal{L}^2$-subspace generated by this family

$$f \in V_0 \quad \Longleftrightarrow \quad \begin{cases} \text{the exists an } \ell^2\text{-sequence } (c_n)_{n\in\mathbb{Z}} \text{ with} \\ \widehat{f}(s) = \widehat{\phi}(s) \cdot \sum_{n\in\mathbb{Z}} c_n e^{-2\pi i n s} \end{cases}$$

  In words:
  the elements of V_0 are precisely those $\mathcal{L}^2$-functions f, whose FT $\widehat{f}$ is a product of $\widehat{\phi}$ and a period-1 Fourier series

- For the proof (not difficult, using Bessel's inequality and Parseval-Plancherel) see the Lecture Notes

- The following Theorem shows how the construction leading to an MRA for the pieceswise-linear functions can be made in a general context. (For the proof see the Lecture Notes)
- Theorem
  - If $\phi(t) \in \mathcal{L}^2(\mathbb{R})$ is a function with compact support
  - and if there exist constants A, B s.th.

$$0 < A \leq \sum_{n \in \mathbb{Z}} \left| \widehat{\phi}(s+n) \right|^2 \leq B,$$

  then there exists a function $\widetilde{\phi}(t) \in \mathcal{L}^2(\mathbb{R})$, such that
  - the family $\left\{ T_k \widetilde{\phi} \right\}_{k \in \mathbb{Z}}$ is an ONST
  - and it generates the same space V_0 as the family $\{ T_k \phi \}_{k \in \mathbb{Z}}$

- General MRA-setup (as before) with
  - scaling function $\phi(t)$, scaling filter $(h_k)_{k\in\mathbb{Z}}$, Fourier series $m_0(s)$
  - wavelet function $\psi(t)$, wavelet filter $(g_k)_{k\in\mathbb{Z}}$, Fourier series $m_1(s)$
- Properties
  1. $\left|\widehat{\phi}(0)\right| = \left|\int_{\mathbb{R}} \phi(t)\,dt\right| = 1$
  2. for all $n \in \mathbb{Z}, n \neq 0$: $\quad \widehat{\phi}(n) = 0$
  3. $\sum_{n\in\mathbb{Z}} \phi(t+n) \equiv 1$
  4. $\widehat{\psi}(0) = \int_{\mathbb{R}} \psi(t)\,dt = 0$
- The proofs are somewhat technical. See the Lecture Notes

Recall properties of the FT w.r.t. smoothness an vanishing at infinity

- Theorem
  - If $f(t) \in \mathcal{L}^1(\mathbb{R})$ and $t \cdot f(t) \in \mathcal{L}^1(\mathbb{R})$, then $\widehat{f}(s) \in \mathcal{C}^1(\mathbb{R})$ and

$$\widehat{t \cdot f}(s) = -\frac{1}{2\pi i}\frac{d}{ds}\widehat{f}(s)$$

  - More generally for $N \geq 1$

    If $f(t) \in \mathcal{L}^1(\mathbb{R})$ and $t^N f(t) \in \mathcal{L}^1(\mathbb{R})$ then $\widehat{f}(s) \in \mathcal{C}^N(\mathbb{R})$ and

$$(\widehat{t^j f(t)})(s) = \left(-\frac{1}{2\pi i}\frac{d}{ds}\right)^j \widehat{f}(s) \quad (0 \leq j \leq N)$$

    "and conversely"

- Note: "$t^N f(t) \in \mathcal{L}^1(\mathbb{R})$" means: $f(t)$ vanishes rapidly as $t \to \pm\infty$, typically $f(t) \in \mathcal{O}(t^{-N-1-\varepsilon})$ for some $\varepsilon > 0$;
  "$\widehat{f}(s) \in \mathcal{C}^N(\mathbb{R})$" means that $\widehat{f}(t)$ has N continuous derivatives

- For function $f(t)$ and $k \geq 0$ the *k-th moment* is defined as

$$\int_{\mathbb{R}} t^k f(t)\, dt$$

- Note: if $t^k f(t) \in \mathcal{L}^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} t^k f(t)\, dt = 0 \quad \Longleftrightarrow \quad \widehat{f}^{(k)}(0) = 0$$

- Theorem

  If $\psi \in \mathcal{L}^2(\mathbb{R})$ and if $\{\psi_{j,k}\}$ is an orthonormal family in $\mathcal{L}^2(\mathbb{R})$, then:
  - If $\psi, \widehat{\psi} \in \mathcal{L}^1(\mathbb{R})$, then $\int_{\mathbb{R}} \psi = 0$
  - More generally: if $t^N \psi(t), s^{N+1}\widehat{\psi}(s) \in \mathcal{L}^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} t^m\, \psi(t)\, dt = \widehat{\psi}^{(m)}(0) = 0 \quad (0 \leq m \leq N)$$

- Remarks
  - If a function $f(t)$ satisfies

$$\widehat{f}^{(k)}(0) = \int_{\mathbb{R}} t^k f(t)\, dt = 0 \quad (0 \leq k < N),$$

    then f is said to have N *vanishing moments*
  - The previous Theorem relates smoothness and vanishing at infinity of a wavelet function $\psi(t)$ with the phenomenon of vanishing moments
  - The FT of the wavelet equation

$$\widehat{\psi}(s) = m_1(s/2) \cdot \widehat{\phi}(s/2)$$

    can be differentiated repeatedly, giving

$$m_0^{(k)}(1/2) = 0 \quad (0 \leq k < N)$$

    as a statement equivalent to

$$\psi(t) \text{ has } N \text{ vanishing moments}$$

- Taking the FT of the scaling identity

$$\widehat{\phi}(s) = m_0(s/2) \cdot \widehat{\phi}(s/2)$$

and differentiating it repeatedly gives

$$\widehat{\phi}^{(k)}(m) = 0 \quad \begin{cases} 0 \le k < N \\ m \in \mathbb{Z} \setminus \{0\} \end{cases}$$

The consequences of a wavelet function $\psi(t)$ having N vanishing moments can be made precise:

- Theorem
  If $\psi \in L^2(\mathbb{R})$ has compact support and N vanishing moments, then for each $f \in \mathcal{C}^N(\mathbb{R})$ with $f^{(N)}$ bounded there exists a constant $C = C(N, f)$ s.th.

$$|\langle f \,|\, \psi_{j,k} \rangle| \leq C \cdot 2^{-jN} \cdot 2^{-j/2} \quad (j, k \in \mathbb{Z})$$

- This quantitative statement should be read qualitatively as: *Wavelet coefficients belonging to regions where f is smooth tend to be very small over many levels of resolution!*
- The proof is by using a Taylor expansion of $f(t)$ in the region where $\psi_{j,k}$ is nonzero — see the Lecture Notes

D_4 as an example

- The wavelet function $\psi(t)$ of the Daubechies D_4 filter has $N = 2$ vanishing moments
- One has

$$\int_{\mathbb{R}} \psi(t)\,dt = 0, \qquad \int_{\mathbb{R}} t\,\psi(t)\,dt = 0, \qquad \int_{\mathbb{R}} t^2\,\psi(t)\,dt = -\frac{1}{8}\sqrt{\frac{3}{2\pi}}.$$

- For $f \in \mathcal{C}^2(\mathbb{R})$, by taking the support of $\psi(t)$ into account,

$$\langle\, f \,|\, \psi_{j,k}\,\rangle = \int_{\mathbb{R}} f(t)\,2^{j/2}\,\psi(2^j t - k)\,dt = \int_0^{3\,2^{-j}} f(t + 2^{-j}k)\,2^{j/2}\,\psi(2^j t)\,dt$$

- Expanding $f(t)$ at $t + 2^{-j}k$ in a Taylor series gives

$$\langle\, f \,|\, \psi_{j,k}\rangle \approx -\frac{1}{16}\sqrt{\frac{3}{2\pi}}\,2^{-5j/2}\,f''(2^{-j}k),$$

with equality (instead of $\approx$) if f is a constant, linear or quadratic polynomial

- In particular: all wavelet coefficients $\langle\, f \,|\, \psi_{j,k}\,\rangle$ vanish for regions where f is linear

Wrapping things up:

- Theorem

  If $\phi(t)$ resp. $\psi(t)$ are scaling resp. wavelet functions of an MRA, $\boldsymbol{h} = (h_n)_{n\in\mathbb{Z}}$ the scaling filter and $m_0(s)$ its Fourier series, then the following statements are equivalent:

  1. ψ has N vanishing moments:

  $$\int_{\mathbb{R}} t^k \psi(t) dt = 0 \quad (0 \leq k < N)$$

  2. The filter $\boldsymbol{h} = (h_n)$ satisfies N low-pass conditions

  $$m_0^{(k)}(1/2) = 0 \quad (0 \leq k < N)$$

  3. The Fourier series $m_0(s)$ of $\boldsymbol{h} = (h_n)$ can be factored:

  $$m_0(s) = (\frac{1 + e^{-2\pi is}}{2})^N L(s)$$

  where $L(s)$ is a period-1 trigonometric polynomial

  4. The QMF $\boldsymbol{h} = (h_n)$ satisfies the N moment conditions

  $$\sum_{n\in\mathbb{Z}} (-1)^n \, h_n \, n^k = 0 \quad (0 \leq k < N)$$