Exercise 9 Solution

We start with the definition of SNR

$$\mathrm{SNR} = \frac{\mathrm{E}\left(f'^2\right)}{\mathrm{E}\left(n^2\right)} \ . \tag{1}$$

Two of the three assumptions from the lecture (slide 4-14) are still valid

1. Zero-mean signal and noise

$$\mathrm{E}\left(f'\right) = \mathrm{E}\left(n\right) = 0$$

2. Quantization error is uniformly distributed

$$\mathrm{p}\left(n\right) = \begin{cases} \frac{1}{s}, & \text{for } -\frac{s}{2} \leq n \leq \frac{n}{2} \\ 0, & \text{otherwise} \end{cases}$$

The third assumption about the range of signal values changes. The assumption in the lecture is that $-4\sigma_{f'} \leq f' \leq 4\sigma_{f'}$. Here, we want to look at the generalized case

$$-k\sigma_{f'} \leq f' \leq k\sigma_{f'} \ . \tag{2}$$

From the lecture (slide 4-17) we know that

$$\mathrm{E}\left(f'^2\right) = \sigma_{f'}^2 \ . \tag{3}$$

With k instead of 4, the quantization step becomes

$$s = \frac{2k\sigma_{f'}}{2^B} \ , \tag{4}$$

where B is the number of quantization bits. Now we determine $\mathrm{E}\left(n^2\right)$ (slide 4-18) as

$$\mathrm{E}\left(n^2\right) = \frac{s^2}{12} = \frac{1}{12}\left(\frac{2k\sigma_{f'}}{2^B}\right)^2 = \frac{2^2}{12}\frac{k^2\sigma_{f'}^2}{2^{2B}} \ . \tag{5}$$

Now we can insert equations 3 and 5 into equation 1

$$\mathrm{SNR} = \frac{\mathrm{E}\left(f'^2\right)}{\mathrm{E}\left(n^2\right)} = \frac{\sigma_{f'}^2}{\frac{2^2}{12}\frac{k^2\sigma_{f'}^2}{2^{2B}}} = \frac{12 \cdot 2^{2B}\sigma_{f'}^2}{2^2k^2\sigma_{f'}^2} = \frac{12}{k^2} \cdot 2^{2B-2} \ . \tag{6}$$

This yields

$$\begin{aligned} \mathrm{SNR_{dB}} &= 10\log_{10}\mathrm{SNR} && (7) \\ &= 10 \cdot \left(\log_{10} 12 - \log_{10} k^2 + \log_{10} 2^{2B-2}\right) && (8) \\ &= 10 \cdot \left(\log_{10} 12 - 10\log_{10} k + (2B-2)\log_{10} 2\right) && (9) \\ &\approx 10.79 - 20\log_{10} k + 6.02B - 6.02 && (10) \\ &\approx 4.77 - 20\log_{10} k + 6.02B && (11) \end{aligned}$$

If $B+1$ bits are used, the SNR increase is approximately 6dB.